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Index Sets and Universal Numberings

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T e c h n i c a l   R e p o r t

Foreword

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Index Sets and Universal Numberings*

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Abstract. This paper studies the Turing degrees of various properties defined for universal numberings, that is, for numberings which list all partial-recursive functions. In particular properties relating to the domain of the corresponding functions are investigated like the set DEQ of all pairs of indices of functions with the same domain, the set DMIN of all minimal indices of sets and DMIN* of all indices which are minimal with respect to equality of the domain modulo finitely many differences. A partial solution to a question of Schaefer is obtained by showing that for every universal numbering with the Kolmogorov property, the set DMIN* is Turing equivalent to the double jump of the halting problem. Furthermore, it is shown that the join of DEQ and the halting problem is Turing equivalent to the jump of the halting problem and that there are numberings for which DEQ itself has 1-generic Turing degree.

1 Introduction

It is known that for acceptable numbering many problems are very hard: Rice [16] showed that all semantic properties like $\{e : \varphi_e \text{ is total}\}$ or $\{e : \varphi_e \text{ is somewhere defined}\}$ are non-recursive and that the halting problem K is Turing reducible to them. Similarly, Meyer [12] showed that the set $\text{MIN}_\varphi = \{e : \forall d < e\, [\varphi_d \neq \varphi_e]\}$ of minimal indices is even harder: $\text{MIN}_\varphi \equiv_T K'$. In contrast to this, Friedberg [5] showed that there is a numbering ψ of all partial-recursive functions such that $\psi_d \neq \psi_e$ whenever $d \neq e$. Hence, every index in this numbering is a minimal index: $\text{MIN}_\psi = \mathbb{N}$. One could also look at the corresponding questions for minimal indices for domains. Then, as long as one does not postulate that every function occurs in the numbering but only that every domain occurs, there are numberings for which the set of minimal indices of domains is recursive and other numberings for which this set is Turing dequivalent to K'. But there is a different result if one requires that the numbering is universal in the sense that it contains every partial-recursive function. Then the set $\text{DMIN}_\psi = \{e : \forall d < e\, [W_d^\psi \neq W_e^\psi]\}$ is not recursive but satisfies

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$\mathrm{DMIN}_\psi \oplus K \equiv_T K'$, see Proposition 4 below. On the other hand, DMIN_ψ is for some universal numberings ψ not above K'. Indeed, DMIN_ψ is 1-generic for certain numberings. In the present work, various properties linked to the domains of functions for universal and domain-universal numberings are studied. In particular the complexities of these sets are compared with K, K', K'' and so on.

Schaefer [17] tried to lift Meyer's result one level up in the arithmetic hierarchy and asked whether $\mathrm{MIN}^*_\psi \equiv_T K''$; Teutsch [19] asked the corresponding question for domains: is $\mathrm{DMIN}^*_\psi \equiv_T K''$? These questions were originally formulated for Gödel numberings. In the present work, partial answers are obtained: on one hand, if the numbering ψ is a Kolmogorov numbering then DMIN^*_ψ and MIN^*_ψ are both Turing equivalent to K''; on the other hand, there is a universal numbering (which is not a Gödel numbering) such that DMIN^*_ψ and MIN^*_ψ are 1-generic and hence not above K.

Besides this, a further main result of this paper is to show that for certain universal numbering ψ the domain equality problem DEQ_ψ has 1-generic Turing degree; hence the domain-equivalence problem of ψ is not Turing hard for K'.

After this short overview of the history of minimal indices and the main results of this paper, the formal definitions are given, beginning with the fundamental notion of numberings and universal numberings. For an introduction to the basic notions of Recursion Theory and Kolmogorov Complexity, see the textbooks of Li and Vitányi [11], Odifreddi [13, 14] and Soare [18].

Definition 1. Let $\psi_0, \psi_1, \psi_2, \ldots$ be a family of functions from $\mathbb{N}$ to $\mathbb{N}$. ψ is called a *numbering* iff the set $\{\langle e, x, y\rangle : \psi_e(x) \downarrow = y\}$ is recursively enumerable; ψ is called a *univeral numbering* iff every partial-recursive function equals to some function ψ_e; ψ is called a *domain-universal numbering* iff for every r.e. set A there is an index e such that the domain W_e^ψ of ψ_e equals A.

A numbering ψ is *acceptable* or a *Gödel numbering* iff for every further numbering ϑ there is a recursive function f such that $\psi_{f(e)} = \vartheta_e$ for all e; a numbering ψ has the *Kolmogorov property* iff

$$\forall \text{ numberings } \vartheta\ \exists c\ \forall e\ \exists d < ce + c\ [\psi_d = \vartheta_e]$$

and a numbering ψ is a *Kolmogorov numbering* iff it has the Kolmogorov property effectively, that is,

$$\forall \text{ numberings } \vartheta\ \exists c\ \exists \text{ recursive } f\ \forall e\ [f(e) < ce + c \wedge \psi_{f(e)} = \vartheta_e].$$

A numbering ψ is a *K-Gödel numbering* [3] iff for every further numbering ϑ there is a K-recursive function f such that $\psi_{f(e)} = \vartheta_e$ for all e. Similarly one can define *K-Kolmogorov numberings.*

Note that a universal numbering is a weakening of an acceptable numbering while in the field of Kolmogorov complexity, the notion of a universal machine is stronger than that of an acceptable numbering; there a universal machine is a numbering of strings (not functions) with the Kolmogorov property and so this notion is stronger than the notion of an acceptable numbering of strings.

Definition 2. Given a numbering ψ, define that $\mathrm{DMIN}_\psi = \{e : \forall d < e\ [W_d^\psi \neq W_e^\psi]\}$, $\mathrm{DMIN}_\psi^* = \{e : \forall d < e\ [W_d^\psi \neq^* W_e^\psi]\}$ and $\mathrm{DMIN}_\psi^m = \{e : \forall d < e\ [W_d^\psi \not\equiv_m W_e^\psi]\}$. Here $A =^* B$ means that the sets A, B are finite variants and $A \neq^* B$ means that the sets A, B are not finite variants. Furthermore, $A \equiv_m B$ iff there are recursive functions f, g such that $A(x) = B(f(x))$ and $B(x) = A(g(x))$ for all x; $A \not\equiv_m B$ otherwise. The superscript "m" in DMIN_ψ^m is just referring to many-one reduction.

2 Minimal Indices and Turing Degrees

The next result is well-known and can, for example, be derived from [4, Theorem 5.7]. The proof below is given for the reader's convenience and not claimed to be novel.

Proposition 3. *Let φ be any acceptable numbering. Now $K' \leq_T A \oplus K$ iff one can enumerate relative to an oracle A a set E of indices of total recursive functions such that for every total recursive f there is an $e \in E$ with $\varphi_e = f$.*

Proof. The two directions of the theorem are proven, one after the other.

On one hand, assume that $K' \leq_T A \oplus K$ and define a recursive function f such that for all indices e and finite sets D it holds that

$$\varphi_{f(e,D)}(x) = \begin{cases} \varphi_e(x) & \text{if there is a stage } t \text{ such that} \\ & \varphi_{e,t}(x) \text{ is defined and } K_t \cap D = \emptyset; \\ 0 & \text{if there is a stage } t \text{ such that} \\ & \varphi_{e,t}(x) \text{ is undefined and } K_{t+1} \cap D \neq \emptyset; \\ \uparrow & \text{otherwise.} \end{cases}$$

Furthermore, let an enumeration of all indices of total functions relative to $A \oplus K$ be given. Now, the new enumeration relative to A is made by enumerating all indices of the form $f(e, D)$ where there is a stage s such that e is output by the original enumeration algorithm using the oracle $A \oplus K_s$ in place of $A \oplus K$ and D is the set of places of K_s queried where the answer was 0.

For the verification of the algorithm, consider first the case that s is so large that an index e is enumerated relative to $A \oplus K_s$ using the original algorithm by the same queries and answers as relative to $A \oplus K$. Then the D obtained satisfies $K \cap D = \emptyset$ and the index $f(e, D)$ produced by the new enumeration relative to A satisfies that $\varphi_{f(e,D)}$ is total and equal to φ_e. Furthermore, all $f(e, D)$ supplied are indices of total functions as either the index e is produced by the original enumeration and φ_e is total or $D \cap K \neq \emptyset$. In both cases, this condition implies that $\varphi_{f(e,D)}$ is total as one of the two first cases in the definition of the function applies. Hence all indices enumerated are for total functions and every total recursive function is covered.

On the other hand, assume now for the reverse direction that there is an A-r.e. set E such that all indices in E are of total functions and every total recursive function has an index in E. Recall that one can enumerate the set $\{e : \varphi_e \text{ is partial}\}$ relative to K and thus also relative to $A \oplus K$ as it is the set of all e for which there is an x such that $\varphi_e(x)$ is undefined. Furthermore, one can enumerate the set $\{e : \varphi_e \text{ is total}\}$ relative to $A \oplus K$ as it is the set of all e for which there is an $e' \in E$ such that $\forall s\, \forall x$ [if $\varphi_{e',s}(x)$ has halted and output a number below s then $\varphi_{e,s}(x)$ has also halted]. This statement can also be checked with the oracle K. Furthermore, for each index e of a total function there exists an index $e' \in E$ of another total function such that $\varphi_{e'}$ majorizes the time which φ_e needs to converge. Hence, the enumeration procedure is correct and the set of all indices of total functions is recursively enumerable relative to $A \oplus K$. As the set of indices of total functions with respect to the acceptable numbering φ is Π^0_2-complete, $K' \leq_T A \oplus K$. □

Meyer [12] showed the next result for Gödel numberings, here it is given for universal numberings; by a well-known result of Friedberg this is false for some domain-universal numberings.

Proposition 4. *For every universal numbering ψ, $K' \leq_T \mathrm{DMIN}_\psi \oplus K$.*

Proof. Let a be the least number such that ψ_a is total and let $g(e) = \min(\mathbb{N} - W^\psi_e)$ whenever the minimum exists. Note that $g(d)$ is defined for all $d \in \mathrm{DMIN}_\psi - \{a\}$. Now one has that ψ_e is total iff $g(d) \in W^\psi_e$ for all $d \in \mathrm{DMIN}_\psi \cap \{0, 1, 2, \ldots, e\} - \{a\}$. This condition can be checked relative to $\mathrm{DMIN}_\psi \oplus K$ and hence one can enumerate all ψ-indices of total-recursive functions. Now it follows from Proposition 3 that $K' \leq_T \mathrm{DMIN}_\psi \oplus K$. □

Schaefer [17] and Teutsch [19, 20] investigated the complexity of DMIN^*_ψ. The next two results generalize their findings from Gödel numberings to domain-universal numberings.

Proposition 5. *For every domain-universal numbering ψ, $K' \leq_T \mathrm{DMIN}^*_\psi \oplus K$.*

Proof. Let ψ be the given numbering and φ be an acceptable numbering. Let σ_x be the x-th string in a recursive bijection from $\mathbb{N}$ to $\mathbb{N}^*$. Let $\sigma_x(y)$ be the member number y of that string and $\sigma_x(y)\uparrow$ if σ_x does not have a member number y.

Now define the following function $\varphi_{g(e,n)}(x)$ according to that of the first two cases which is found to apply first; the third case is taken if neither the first nor the second case applies:

$$\varphi_{g(e,n)}(x) = \begin{cases} \sigma_a(x) & \text{if } a \in W^\psi_e \wedge a > n \wedge \sigma_a(x)\downarrow; \\ 0 & \text{if there are } b, c \in W^\psi_e \text{ and } y \text{ with } n < b,\ n < c \text{ and } \sigma_b(y)\downarrow \neq \sigma_c(y)\downarrow; \\ \uparrow & \text{otherwise.} \end{cases}$$

The second line in this case-distinction is included to ensure that $\varphi_{g(e,n)}$ is total whenever W^ψ_e is infinite. Let d be the unique index in DMIN^*_ψ such that W^ψ_d is finite. Then $\varphi_{g(e,n)}$ is total for every $e \in \mathrm{DMIN}^*_\psi - \{d\}$ and n. Furthermore, for every recursive f there is an $e \in \mathrm{DMIN}^*_\psi - \{d\}$ such that

$$W^\varphi_e =^* \{a : \exists n\ [\sigma_a = f(0)f(1)f(2)\ldots f(n)]\}.$$

It follows that $\varphi_{g(e,n)} = f$ for almost all n. Hence the set

$$E = \{g(e, n) : e \in \mathrm{DMIN}^*_\psi - \{d\}, n \in \mathbb{N}\}$$

is a set of φ-indices which contains an index for every total recursive function and which consists only of indices of total-recursive functions. It follows from Proposition 3 that $K' \leq_T \mathrm{DMIN}^*_\psi \oplus K$. □

Proposition 6. *For every domain-universal numbering ψ, $K'' \equiv_T \mathrm{DMIN}^*_\psi \oplus K'$.*

Proof. Let φ be a Gödel numbering and note that

$$K'' \equiv_T \{e : W^\varphi_e \text{ is co-finite}\}.$$

Furthermore, let a be the unique element of DMIN^*_ψ such that W^ψ_a is co-finite. For any given e, find using K' the least d such that $W^\psi_d = W^\varphi_e$. Furthermore, let $D = \mathrm{DMIN}^*_\psi \cap \{0, 1, 2, \ldots, d\}$ and define

$$\mathrm{ndiff}(c, d, x) = |\{y \leq x : W^\psi_c(y) \neq W^\psi_d(y)\}|$$

to be the number of differences between W^ψ_c and W^ψ_d below x. One can find with oracle K' the unique $b \in D$ such that $\mathrm{ndiff}(b, d, x) \leq \mathrm{ndiff}(c, d, x)$ for all $c \in D$ and almost all x. Note that b is the unique member of DMIN^*_ψ with $W^\psi_b =^* W^\varphi_e$. Now W^φ_e is co-finite iff $b = a$; hence

$$\{e : W^\varphi_e \text{ is co-finite}\} \leq_T \mathrm{DMIN}^*_\psi \oplus K'.$$

As $\mathrm{DMIN}^*_\psi \leq_T K''$ for all ψ, $K'' \equiv_T \mathrm{DMIN}^*_\psi \oplus K'$. $\square$

Remark 7. The following proofs make use of Owings' Cardinality Theorem [15]. This says that whenever there is an $m > 0$ and a B-recursive $\{0, 1, 2, \ldots, m\}$-valued function mapping every m-tuple $(a_1, a_2, \ldots, a_m)$ to a number in $\{0, 1, 2, \ldots, m\}$ which is different from $A(a_1) + A(a_2) + \ldots + A(a_m)$ then $A \leq_T B$. Kummer [6, 9] generalized this result and showed that whenever there are an $m > 0$ and B-r.e. sets enumerating uniformly for every m-tuple $(a_1, a_2, \ldots, a_m)$ up to m numbers including $A(a_1) + A(a_2) + \ldots + A(a_m)$ then $A \leq_T B$.

Theorem 8. *For every universal numbering ψ with the Kolmogorov property, $K \leq_T \mathrm{DMIN}^*_\psi$ and $K'' \equiv_T \mathrm{DMIN}^*_\psi$.*

Proof. Again let σ_n be the n-th finite string in an enumeration of $\mathbb{N}^*$. Due to the Kolmogorov property, one can recursively partition the natural numbers into intervals I_n such that for every n there is a number $z \in \mathrm{DMIN}^*_\psi$ with $\min(I_n) \cdot (|\sigma_n| + 1) + |\sigma_n| < z < \max(I_n)$. For every $p \in I_n$ with $\sigma_n = a_1 a_2 \ldots a_m$ let

$$\vartheta_p(x) = \psi_{p \cdot (m+1) + K_x(a_1) + K_x(a_2) + \ldots + K_x(a_m)}(x)$$

and note that

$$\vartheta_p =^* \psi_{p \cdot (m+1) + K(a_1) + K(a_2) + \ldots + K(a_m)}$$

as the approximations $K_x(a_1), K_x(a_2), \ldots, K_x(a_m)$ coincide respectively with $K(a_1), K(a_2), \ldots, K(a_m)$ for almost all x. By the Kolmogorov property there is a constant m such that for every p there is an $e < \max\{pm, m\}$ with $\psi_e = \vartheta_p$; fix this m from now on.

Now, for any $a_1, a_2, \ldots, a_m$, choose n such that $\sigma_n = a_1 a_2 \ldots a_m$ and let $g(a_1, a_2, \ldots, a_m)$ and $h(a_1, a_2, \ldots, a_m)$ be the unique values in $\mathbb{N}$ and $\{0, 1, 2, \ldots, m\}$, respectively, such that

$$g(a_1, a_2, \ldots, a_m) \cdot (m + 1) + h(a_1, a_2, \ldots, a_m) = \max(I_n \cap \mathrm{DMIN}^*_\psi).$$

By choice of m, $g(a_1, a_2, \ldots, a_m) \in I_n$ and $g(a_1, a_2, \ldots, a_m) > 0$. Hence

$$\vartheta_{g(a_1,a_2,\ldots,a_m)} =^* \psi_{g(a_1,a_2,\ldots,a_m)\cdot(m+1)+K(a_1)+K(a_2)+\ldots+K(a_m)}$$

and $\psi_e = \vartheta_{g(a_1,a_2,\ldots,a_m)}$ for some $e < g(a_1, a_2, \ldots, a_m) \cdot m$. So $g(a_1, a_2, \ldots, a_m) \cdot (m+1) + K(a_1) + K(a_2) + \ldots + K(a_m)$ is not in DMIN^*_ψ and

$$h(a_1, a_2, \ldots, a_m) \in \{0, 1, 2, \ldots, m\} - \{K(a_1) + K(a_2) + \ldots + K(a_m)\}.$$

So $h \leq_T \mathrm{DMIN}^*_\psi$ and h produces on input $a_1, a_2, \ldots, a_m$ a value in $\{0, 1, 2, \ldots, m\}$ different from $K(a_1) + K(a_2) + \ldots + K(a_m)$. Owings' Cardinality Theorem [6, 15] states that the existence of such a function h implies $K \leq_T \mathrm{DMIN}^*_\psi$.

It is well-known that $\mathrm{DMIN}^*_\psi \leq_T K''$. On the other hand one can now apply Proposition 5 to get that $K' \leq_T \mathrm{DMIN}^*_\psi$ and Proposition 6 to get that $K'' \leq_T \mathrm{DMIN}^*_\psi$. $\square$

Theorem 9. *For every universal numbering ψ with the Kolmogorov property, $K'' \equiv_T \mathrm{DMIN}^m_\psi$.*

Proof. The first part is to show that $K' \leq_T \mathrm{DMIN}^m_\psi$. This is done by applying Owings' Cardinality Theorem for the set $\{a : |W^\varphi_a| = \infty\}$ where φ is an acceptable numbering. The proof is quite similar to the proof of Theorem 8. Again let σ_n be the n-th finite string in an enumeration of $\mathbb{N}^*$ and let $a_1, a_2, \ldots, a_m$ be the numbers with $\sigma_n = a_1 a_2 \ldots a_m$ and let k range over $1, 2, \ldots, m$. Due to the Kolmogorov property, one can recursively partition the natural numbers into intervals I_n such that for every n there is a number $x \in \mathrm{DMIN}^m_\psi$ with $\min(I_n)\cdot(|\sigma_n|+1)+|\sigma_n| < x < \max(I_n)$. Define a numbering ϑ such that, for every n, for $m = |\sigma_n|$ and for every $p \in I_n$, the condition

$$\begin{aligned} W^\vartheta_p = \{(m+1)x + b : b < |\{k \in \{0, 1, 2, \ldots, m\} : |W^\varphi_{a_k}| \geq x\}| \ \vee \\ (b = |\{k \in \{0, 1, 2, \ldots, m\} : |W^\varphi_{a_k}| \geq x\}| \wedge x \in W^\psi_{(m+1)p+b})\} \end{aligned}$$

is satisfied. The goal of this construction is that $W^\psi_{(m+1)p+|\{k\in\{0,1,2,\ldots,m\}:|W^\varphi_{a_k}|=\infty\}|}$ is either recursive or many-one equivalent to W^ϑ_p. To see this, let

$$z = |\{k \in \{0, 1, 2, \ldots, m\} : |W^\varphi_{a_k}| = \infty\}|$$

and

$$y = \min\{x : \forall k \in \{0, 1, 2, \ldots, m\}\, [|W^\varphi_{a_k}| \geq x \Rightarrow |W^\varphi_{a_k}| = \infty\}].$$

Now one has for all $x \geq y$ and $b \in \{0, 1, 2, \ldots, m\}$ that

$$(m+1)x + b \in W^\vartheta_p \Leftrightarrow b < z \vee (b = z \wedge x \in W^\psi_{(m+1)p+z}).$$

It is easy to see that $W^\vartheta_p \equiv_m W^\psi_{(m+1)p+z}$ whenever both sets are neither $\emptyset$ nor $\mathbb{N}$; this is in particular satisfied if $W^\psi_{(m+1)p+z}$ is not recursive.

Now fix m as a number which is so large that three indices of recursive sets in DMIN^m_ψ are in some I_n with $|\sigma_n| < m$ and that for every $p > 0$ there is an index $e < pm$ with $W^\psi_e = W^\vartheta_p$. Given

$a_1, a_2, \ldots, a_m$, let n be the index with $\sigma_n = a_1 a_2 \ldots a_m$ and define the values $g(a_1, a_2, \ldots, a_m) \in \mathbb{N}$ and $h(a_1, a_2, \ldots, a_m) \in \{0, 1, 2, \ldots, m\}$ such that

$$g(a_1, a_2, \ldots, a_m) \cdot (m+1) + h(a_1, a_2, \ldots, a_m) = \max(\mathrm{DMIN}^m_\psi \cap I_n).$$

From the choice of the intervals it follows that $g(a_1, a_2, \ldots, a_m) \in I_n$ and

$$h(a_1, a_2, \ldots, a_m) \neq |\{k \in \{0, 1, 2, \ldots, m\} : |W^\varphi_{a_k}| = \infty\}|$$

as $W^\psi_{(m+1)g(a_1,a_2,\ldots,a_m)+|\{k\in\{0,1,2,\ldots,m\}:|W^\varphi_{a_k}|=\infty\}|}$ is either recursive or many-one equivalent to a set with a smaller index. Using Owings' Cardinality Theorem [15], one obtains that

$$K' \equiv_T \{a : |W^\varphi_a| = \infty\} \leq_T \mathrm{DMIN}^m_\psi.$$

The index set $\{e : W^\varphi_e \text{ is recursive}\}$ has the same Turing degree as K''. One can use the oracle K' in order to find for given e the corresponding d such that $W^\psi_d = W^\varphi_e$ and then one can determine $D = \mathrm{DMIN}^m_\psi \cap \{0, 1, 2, \ldots, d\}$. Using the oracle K' one can find the unique member of D which is many-one equivalent to W^ψ_d and compare it to the minimal indices of the three recursive many-one degrees. It follows that

$$\{e : W^\varphi_e \text{ is recursive }\} \leq_T \mathrm{DMIN}^m_\psi$$

and, using $\mathrm{DMIN}^m_\psi \leq_T K''$, one gets $\mathrm{DMIN}^m_\psi \equiv_T K''$. ☐

Remark 10. Teutsch [19, 20] considered also the problem $\mathrm{DMIN}^T_\psi = \{e : \forall d < e\, [W^\psi_d \not\equiv_T W^\psi_e]\}$. He showed that if ψ is an acceptable numbering then $K''' \leq_T \mathrm{DMIN}^T_\psi \oplus K'$. The above techniques can also be used to show that if ψ is a Kolmogorov numbering then $K''' \equiv_T \mathrm{DMIN}^T_\psi$.

One might also ask what the minimum Turing degree of MIN^*_ψ is. While Friedberg showed that MIN_ψ can be recursive, this is not true for MIN^*_ψ. Indeed, one can easily adapt the construction from Theorem 19 to construct a K-Gödel numbering where MIN^*_ψ is 1-generic. In addition to this, Propositions 5 and 6 can be transferred to MIN^*_ψ. Furthermore, the proof from above for numberings with the Kolmogorov property works also for MIN^*_ψ instead of DMIN^*_ψ. Thus one has the following result.

Theorem 11. *For every universal numbering ψ with the Kolmogorov property, $K' \equiv_T \mathrm{MIN}_\psi$, $K'' \equiv_T \mathrm{MIN}^*_\psi$ and $K'' \equiv_T \mathrm{MIN}^m_\psi$.*

3 Prominent Index Sets

It is known from Rice's Theorem that almost all index sets in Gödel numberings are Turing hard for K. On the other hand, in Friedberg numberings, the index set of the everywhere undefined function is just a singleton and hence recursive. So it is a natural question how the index sets depend on the chosen underlying universal numbering. In particular the following index sets are investigated within this section.

Definition 12. For a universal numbering ψ define the following notions:

- $\mathrm{EQ}_\psi = \{\langle i, j\rangle : \psi_i = \psi_j\}$ and $\mathrm{EQ}^*_\psi = \{\langle i, j\rangle : \psi_i =^* \psi_j\}$;
- $\mathrm{DEQ}_\psi = \{\langle i, j\rangle : W^\psi_i = W^\psi_j\}$ and $\mathrm{DEQ}^*_\psi = \{\langle i, j\rangle : W^\psi_i =^* W^\psi_j\}$;
- $\mathrm{INC}_\psi = \{\langle i, j\rangle : W^\psi_i \subseteq W^\psi_j\}$ and $\mathrm{INC}^*_\psi = \{\langle i, j\rangle : W^\psi_i \subseteq^* W^\psi_j\}$;
- $\mathrm{EXT}_\psi = \{\langle i, j\rangle : \forall x \in W^\psi_i\ [x \in W^\psi_j \wedge \psi_j(x) = \psi_i(x)]\}$;
- $\mathrm{CONS}_\psi = \{\langle i, j\rangle : \forall x \in W^\psi_i \cap W^\psi_j\ [\psi_i(x) = \psi_j(x)]\}$;
- $\mathrm{DISJ}_\psi = \{\langle i, j\rangle : W^\psi_i \cap W^\psi_j = \emptyset\}$;
- $\mathrm{INF}_\psi = \{i : W^\psi_i \text{ is infinite}\}$.

Note that although these sets come as a sets of pairs, one can also fix the index i and consider the classic index set of all j such that ψ_j is consistent with ψ_i in the way described above. But the index sets of pairs are quite natural and so some more of these examples will be investigated.

Kummer [10] obtained a breakthrough and solved an open problem of Herrmann posed around 10 years earlier by showing that there is a domain-universal numbering where the domain inclusion problem is K-recursive. He furthermore concluded that also the extension-problem for universal numberings can be made K-recursive.

Theorem 13 (Kummer [10]). *There is a domain universal numbering ψ and a universal numbering ϑ such that*

- $\mathrm{INC}_\psi \leq_T K$;
- $\mathrm{EXT}_\vartheta \equiv_T K$.

The numbering ϑ can easily be obtained from ψ.

Note that this result needs that ψ is only domain universal and not universal; if ψ would be universal then $K' \leq_T \mathrm{INC}_\psi \oplus K$ and hence $\mathrm{INC}_\psi \not\leq_T K$. It is still open whether $K \leq_T \mathrm{INC}_\psi$ for all domain universal numberings ψ. But for the function-extension problem, Kummer's result is optimal.

Proposition 14. $\mathrm{EXT}_\psi \geq_T K$ *for every universal numbering.*

Proof. Let $a_0, a_1, a_2, \ldots$ be a recursive enumeration of K and choose i such that $\psi_i(x)$ is the least s with $a_s = x$ whenever such an s exists, that is, whenever $x \in K$. Now one can compute $K(x)$ by using the oracle EXT_ψ to search for a j where $\psi_j(x)$ is defined and $\langle i, j\rangle \in \mathrm{EXT}_\psi$. This j exists since it can be obtained by modifying the function ψ_i just at x in the case that $\psi_i(x)$ is undefined. Now $x \in K$ iff $a_{\psi_j(x)} = x$: if $x \in K$ then $\psi_j(x) = \psi_i(x)$ and $x = a_{\psi_i(x)}$ by definition; if $x \notin K$ then $x \notin \{a_0, a_1, a_2, \ldots\}$ and therefore $x \neq a_{\psi_j(x)}$. Hence $K \leq_T \mathrm{EXT}_\psi$. □

As Kummer showed, this result cannot be improved. But in the special case of K-Gödel numberings, EXT_ψ takes the Turing degree of K' as shown in the next result.

Theorem 15. $\mathrm{EXT}_\psi \equiv_T K'$ *for every K-Gödel numbering ψ.*

Proof. Let ψ be a given K-Gödel numbering. Clearly $\mathrm{EXT}_\psi \leq_T K'$.

Furthermore, by Proposition 14, $K \leq_T \mathrm{EXT}_\psi$. Now, using this result, it is shown that $K' \leq_T \mathrm{EXT}_\psi$. Let j be the index of the partial-recursive function which satisfies, for some Gödel numbering φ, that

$$\psi_j(\langle e,t\rangle) = \begin{cases} 0 & \text{if } t \leq |W_e^\varphi|; \\ \uparrow & \text{if } t > |W_e^\varphi|. \end{cases}$$

As ψ is K-acceptable, one can now, given any e, using the oracle EXT_ψ, find an index i such that $\psi_i(\langle e,t\rangle) = 0$ for all t and ψ_i is undefined at all other places. Then W_e^φ is infinite iff ψ_j extends ψ_i, that is, if $\langle i,j\rangle \in \mathrm{EXT}_\psi$. Hence $\{e : |W_e^\varphi| < \infty\} \leq_T \mathrm{EXT}_\psi$. This completes the proof of $\mathrm{EXT}_\psi \geq_T K'$. $\square$

The next result is not that difficult and proves that there is one index set whose Turing degree is independent of the underlying numbering: the index set of the consistent functions. One direction can easily be seen as CONS_ψ is co-r.e. and the other direction follows by virtually the same proof as Proposition 14.

Proposition 16. $\mathrm{CONS}_\psi \equiv_T K$ *for all universal numberings* ψ*.*

Remark 17. Another example of this type is the set DISJ_ψ. Here $\mathrm{DISJ}_\psi \equiv_T K$ for every domain-universal numbering ψ. The sufficiency is easy as one can test with one query to the halting problem whether W_i^ψ and W_j^ψ intersect. The necessity is done by showing that the complement of K is r.e. relative to DISJ_ψ: Let i be an index with $W_i^\psi = K$. Then $x \notin K$ iff there is an j with $x \in W_j^\psi \wedge \langle i,j\rangle \in \mathrm{DISJ}_\psi$. Hence the complement of K is recursively enumerable relative to DISJ_ψ and so $K \leq_T \mathrm{DISJ}_\psi$.

Tennenbaum defined that A *is Q-reducible to* B [13, Section III.4] iff there is a recursive function f with $x \in A \Leftrightarrow W_{f(x)} \subseteq B$ for all x. Again let i be an index of K: $K = W_i^\psi$. Furthermore, define f such that $W_{f(x)} = \{\langle i,j\rangle : x \in W_j^\psi\}$. Now K is Q-reducible to the complement of DISJ_ψ as $x \in K$ iff $W_{f(x)}$ is contained in the complement of DISJ_ψ.

For wtt-reducibility and other reducibilities stronger than wtt, no such result is possible. Indeed, one can choose ψ such that $\{e : \psi_e \text{ is total}\}$ is hypersimple and $W_e^\psi = \emptyset$ iff $e = 0$. Then $\{\langle i,j\rangle : i > 0 \wedge j > 0 \wedge \langle i,j\rangle \in \mathrm{DISJ}_\psi\}$ is hyperimmune and wtt-equvialent to DISJ_ψ. Thus no set wtt-reducible to DISJ_ψ has diagonally nonrecursive wtt-degree [7]. In particular, $K \not\leq_{wtt} \mathrm{DISJ}_\psi$ for this numbering ψ.

Remark 18. Since $\mathrm{DISJ}_\psi \equiv_T K$ for all universal numberings ψ, one might ask whether there are also index sets which are Turing equivalent to K' for all universal numberings. One candidate might be INC_ψ, but this problem is open. For the set $A = \{\langle i,j,k\rangle : W_i^\psi \cap W_j^\psi \subseteq W_k^\psi\}$, it can be proven that $A \equiv_T K'$. Note that $A \leq_T K'$. One can retrieve from A whether a set W_e^ψ is total by asking whether the intersection of $\mathbb{N}$ with itself is contained in W_e^ψ. So it follows from Proposition 3 that $K' \leq_T A \oplus K$. Furthermore, $\mathbb{N} - K$ is recursively enumerable relative to A as $x \notin K$ iff there is an index e such that $x \in W_e^\psi$ and $W_e^\psi \cap K$ is empty. Hence $K \leq_T A$ and thus $A \equiv_T K'$.

Recall that a set A is 1-generic iff for every r.e. set B of strings there is an n such that either $A(0)A(1)A(2)\dots A(n) \in B$ or $A(0)A(1)A(2)\dots A(n)\cdot\{0,1\}^*$ is disjoint from B. Jockusch [8] gives an overview on 1-generic sets.

Theorem 19. *There is a K-Gödel numbering ψ such that*

- DMIN_ψ *and* DMIN^*_ψ *are 1-generic;*
- DEQ_ψ *and* INC_ψ *have the Turing degree K';*
- DEQ^*_ψ *and* INC^*_ψ *have the Turing degree K''.*

Proof. The basic idea is the following: One partitions the natural numbers in the limit into two types of intervals: coding intervals $\{e_m\}$ and genericity intervals J_m. The coding intervals contain exactly one element while the genericity intervals are very large. They satisfy the following requirements:

- $|J_m| \geq c_K(m)$ where c_K is the convergence module of K, that is, where $c_K(m) = \min\{s \geq m : \forall n \leq m\ [n \in K \Rightarrow n \in K_s]\}$. In the construction, an approximation c_{K_s} of c_K from below is used.
- There is a limit-recursive function $m \mapsto \sigma_m$ such that $\sigma_m \in \{0,1\}^{|J_m|}$ and for every $\tau \in \{0,1\}^{\min(J_m)}$ and for every genericity requirement set R_n with $n \leq m$ the following implication holds: if $\tau\sigma_m$ has an extension in R_n then already $\tau\sigma_m \in R_n$. Here

$$R_n = \{\rho \in \{0,1\}^* : \text{ some prefix of } \rho \text{ is enumerated into } W_n^\varphi \text{ within } |\rho| \text{ steps}\}.$$

  Note that the R_n are uniformly recursive and φ is the default Gödel numbering.
- There are infinitely many genericity intervals J_m such that for all $x \in J_m$ it holds that $\sigma_m(x - \min(J_m)) = \mathrm{DMIN}_\psi(x) = \mathrm{DMIN}^*_\psi(x)$.

All strings $\sigma_{k,0}$ are just 0 and in stage $s+1$ the following is done:

- Inductively over k define $e_{0,s} = 0$ and $e_{k+1,s} = e_{k,s} + |\sigma_{k,s}| + 1$ and $J_{k,s} = \{x : e_{k,s} < x < e_{k+1,s}\}$.
- Determine the minimal m such that one of the following three cases hold:
  (ρ_m) $m < s$ and $\exists \rho_m \in \{0,1\}^s\, \exists \tau \in \{0,1\}^{\min(J_{m,s})}\, \exists n \leq m\, [\tau\sigma_{m,s}\rho_m \in R_n \wedge \tau\sigma_{m,s} \notin R_n]$;
  (c_K) $m < s$ and $|J_{m,s}| < c_{K_s}(m) \leq s$;
  (none) $m = s$.
  Note that one of the three cases is always satisfied and thus the search terminates.
- In the case (ρ_m), update the approximations to σ_m as follows:

$$\sigma_{k,s+1} = \begin{cases} \sigma_{k,s}\rho_m & \text{if } k = m; \\ \sigma_{k,s} & \text{if } k \neq m. \end{cases}$$

- In the case (c_K) the major goal is to make the interval $J_{m,s}$ having a sufficient long length. Thus

$$\sigma_{k,s+1} = \begin{cases} \sigma_{k,s}0^s & \text{if } k = m; \\ \sigma_{k,s} & \text{if } k \neq m. \end{cases}$$

– In the case (none), no change is made, that is, $\sigma_{k,s+1} = \sigma_{k,s}$ for all k.

Let e_m, J_m, σ_m be the limit of all $e_{m,s}, J_{m,s}, \sigma_{m,s}$. One can show by induction that all these limits exists. The set $\{d : \exists m\ [d \in J_m]\}$ is recursively enumerable as whenever $e_{m,s+1} \neq e_{m,s}$ then $e_{m,s+1} \geq s$; hence $\exists m\ [d \in J_m]$ iff $\exists s > d+1\ \exists m\ [d \in J_{m,s}]$. Now one constructs the numbering ψ from a given universal numberings φ by taking for any d, x the first case which is found to apply:

– if there are $s > x + d$ and $m \leq d$ with $d = e_{m,s}$ and $\varphi_{m,s}(x)$ defined then let $\psi_d(x) = \varphi_m(x)$;
– if there are $s > x + d$ and $m \leq d$ with $d \in J_{m,s}$ and $(\sigma_{m,s}(d - \min(J_m)) = 0) \vee \forall y\ [x \neq \langle d, y\rangle]$ then let $\psi_d(x) = 0$;
– if none of these two cases ever applies then $\psi_d(x)$ remains undefined.

Without loss of generality it is assumed that φ_0 is total and thus 0 is the least index e with $W_e^\psi = \mathbb{N}$. It is easy to see that the following three constraints are satisfied.

– If $d = e_m$ then $\psi_d = \varphi_m$;
– If $d \in J_m$ and $\sigma_m(d - \min(J_m)) = 1$ then $W_d^\psi =^* \{\langle x, y\rangle : x \neq d\}$;
– If $d \in J_m$ and $\sigma_m(d - \min(J_m)) = 0$ then $W_d^\psi = \mathbb{N}$.

Note that the first condition is co-r.e.: Hence one can either compute from d an m with $e_m = d$ or find out that d is in $\bigcup_{m\in\mathbb{N}} J_m$. But it might be that one first comes up with a candidate m for $e_m = d$ and later finds out that actually $e_m \in \cup_m J_m$. So the algorithm is first to determine an m and to follow φ_m where m is correct whenever really the first case applies; later, in the case that the second or third case applies, one has already fixed finitely many values of ψ_d which does not matter as the second and third case tolerate finitely many fixed values. Indeed, one knows only in the limit whether the second or the third case will apply. Therefore only $=^*$ is postulated in the second case and ψ_d is made total in the third case by making $\psi_d(x)$ defined whenever there is an $s > x$ such that $d \in J_{m,s}$ and $\sigma_{m,s}(d - \min(J_{m,s})) = 0$.

For each n there is at most one interval J_m and at most one $d \in J_m$ such that $d > e_n$ and $W_d^\psi =^* \{\langle x, y\rangle : x \neq d\} =^* W_{e_n}^\psi$; if d exists then let $F(n) = d$ else let $F(n) = 0$. Now for every J_m and every $d \in J_m$, $d \in \mathrm{DMIN}_\psi^*$ iff $\sigma_m(d - \min(J_m)) = 1$ and $d \neq F(n)$ for all $n \leq m$. As there are infinitely many indices of total functions, $F(m) = 0$ infinitely often and there are infinitely many genericity intervals J_m which do not intersect the range of F. For each such interval J_m and every d not in the range of F, the construction of σ_m and ψ implies the following: if $\sigma_m(d - \min(J_m)) = 1$ then $d \in \mathrm{DMIN}_\psi \cap \mathrm{DMIN}_\psi^* \wedge W_d^\psi \neq^* \mathbb{N}$ else $d \notin \mathrm{DMIN}_\psi \cup \mathrm{DMIN}_\psi^* \wedge W_d^\psi = \mathbb{N}$. Thus if τ is the characteristic function of DMIN_ψ or DMIN_ψ^* restricted to the domain $\{0, 1, 2, \ldots, e_m\}$ and $n \leq m$ then $\tau\sigma_m \in R_n$ whenever some extension of $\tau\sigma_m$ is in R_n. Hence the sets DMIN_ψ and DMIN_ψ^* are both 1-generic.

Furthermore, let $\{a_0, a_1, a_2, \ldots\}$ be either $\{d : W_d^\psi = \{0\}\}$ or $\{d : W_d^\psi =^* \{0\}\}$. It is easy to see that $\{a_0, a_1, a_2, \ldots\} \subseteq \{e_0, e_1, e_2, \ldots\}$ and $a_n \geq e_n$. By construction $a_{n+1} \geq e_{n+1} \geq c_K(n)$ for all n and it follows that $K \leq_T \mathrm{DEQ}_\psi$, $K \leq_T \mathrm{DEQ}_\psi^*$, $K \leq_T \mathrm{INC}_\psi$ and $K \leq_T \mathrm{INC}_\psi^*$. Having the oracle K and knowing that ψ is a K-Gödel numbering, one can now use the same methods as in Gödel numberings to prove that the sets DEQ_ψ and INC_ψ (respectively, DEQ_ψ^* and INC_ψ^*), are complete for K' (respectively, complete for K''). $\square$

Remark 20. Note that the proof of the above theorem also shows that the set $\{e : \psi_e \text{ is total}\}$ is 1-generic. Using Sacks' Splitting Theorem iteratively, it can be shown [1, 2] that one can produce an uniformly r.e. array of disjoint r.e. sets $A_0, A_1, A_2, \ldots$ such that $A_i \not\leq_T A_j$ whenever $i \neq j$. Now one keeps the construction of Theorem 19 the same until one reaches the construction of W_d^ψ which is now done as follows:

- If $d = e_m$ then $\psi_d = \varphi_m$;
- If $d \in J_m$ and $\sigma_m(d - \min(J_m)) = 1$ then W_d^ψ is a finite variant of A_d;
- If $d \in J_m$ and $\sigma_m(d - \min(J_m)) = 0$ then W_d^ψ is finite.

One can verify using the remaining part of the proof of Theorem 19 that the numbering ψ satisfies that DMIN_ψ^m and DMIN_ψ^T are 1-generic. Hence these two sets are not above K.

In contrast to this result, it can never happen that MIN_ψ is 1-generic.

Proposition 21. *For any universal numbering ψ, the set MIN_ψ is never 1-generic and never hyperimmune.*

Proof. Jockusch and Posner [18, Exercise VI.3.8] noted that 1-generic sets are hyperimmune; see also Jockusch's overview [8] of the degrees of generic sets. Hence it is enough to show that MIN_ψ is not hyperimmune. So let $f(n)$ be the first number s found such that for all $m \leq n$ there is an index $e_m \leq s$ with $\psi_{e_m,s}(0) = m$. This bound s exists since every constant function has an index in the ψ-numbering and thus the search terminates. Now one knows that all function ψ_{e_m} are different and hence there are $n+1$ different functions below $f(n)$. It follows that $|\text{MIN}_\psi \cap \{0, 1, 2, \ldots, f(n)\}| > n$ for all n and hence MIN_ψ is not hyperimmune. ☐

Proposition 22. *There is a domain-universal numbering η such that every infinite r.e. set equals to exactly one W_e^η and INF_η is 1-generic.*

Proof. First, for a given r.e. set E to be determined later, let $\phi_0, \phi_1, \phi_2, \ldots$ be a one-one numbering of all functions with range 0 for which the domain is either $\mathbb{N}$ or has at least 2 non-elements or has the form $\mathbb{N} - \{e\}$ for some $e \in E$: starting with a recursive one-one numbering $u_0, u_1, u_2, \ldots$ of E and a domain-universal Friedberg numbering $\phi'_0, \phi'_1, \phi'_2, \ldots$ in which each domain occurs exactly once, choose ϕ such that

$$
\begin{aligned}
W_0^\phi &= \mathbb{N}, \\
W_{2k+1}^\phi &= \mathbb{N} - \{u_k\} \text{ and} \\
W_{2\langle i,j,k\rangle+2}^\phi &= \{x : x < i\} \cup \{y + i + 1 : y < j\} \cup \{z + i + j + 2 : z \in W_k^{\phi'}\}
\end{aligned}
$$

for any $i, j, k \in \mathbb{N}$.

Second, the idea is now to go on by making a construction as in Theorem 19 with e_m and J_m be defined as there. At the place where W_d^ψ is defined, one defines instead W_d^η by the following adjusted conditions:

- If $d = e_m$ then $\eta_d = \phi_m$;

– If $d \in J_m$ and $\sigma_m(d - \min(J_m)) = 1$ then W_d^η is $\mathbb{N} - \{\langle d, s\rangle\}$ for the first stage s where J_m, σ_m have converged to their final values;
– If $d \in J_m$ and $\sigma_m(d - \min(J_m)) = 0$ then W_d^η is $\{0, 1, 2, \ldots, \langle d, s\rangle - 1\}$ for the first stage s where J_m, σ_m have converged to their final values.

A pair $\langle d, s\rangle$ is enumerated into E iff there is no m such that $d \in J_m$, $\sigma_m(d - \min(J_m)) = 1$ and s is the first stage such that J_m and σ_m have converged to their final values. One can show that E is recursively enumerable and hence one can build the corresponding numbering.

It can be seen that every infinite set V equals to exactly one set W_d^η. Either $V = W_m^\phi$ for some m and then $V = W_{e_m}^\eta$ or $V = \mathbb{N} - \{\langle d, s\rangle\}$ for some pair $\langle d, s\rangle$ not enumerated into E and then $V = W_d^\eta$. So the numbering $W_0^\eta, W_1^\eta, W_2^\eta, \ldots$ contains every infinite set exactly once. The finite sets are contained at least once by the assumption on ϕ but might occur more often. Furthermore, by the choice of J_m and σ_m in Theorem 19, it follows that INF_η is 1-generic. □

Theorem 23. *Assume that the numbering η contains for each infinite r.e. set exactly one index. Then there is a universal numbering ψ with $\mathrm{DEQ}_\psi \equiv_T \mathrm{INF}_\eta$.*

Proof. In the following, let σ_k be the k-th string in a recursive one-one enumeration of all strings with σ_0 being the empty string. Given η, define $\phi_j(x)$ by the first case which is found to apply:

– $\phi_j(x) = 0$ if $|W_j^\eta| > x + 1$ and there are inconsistent strings σ_h, σ_k with $h, k \in W_j^\eta$;
– $\phi_j(x) = \sigma_k(x)$ if $|W_j^\eta| > x + 1$ and k is the first number found in W_j^η with $\sigma_k(x)\downarrow$.

If no case applies then $\phi_j(x)$ is undefined.

Note that a set of at least $x + 2$ strings either contains two incomparable strings or a string of length $x + 1$ or more which is then defined at the input x. Hence whenever $|W_j^\eta| > x + 1$ then ϕ_j is defined by one of the two cases. So, if $j \in \mathrm{INF}_\eta$ then ϕ_j is total else ϕ_j has a finite domain.

Furthermore, for every recursive function f there is a j with $W_j^\eta = \{k : \exists n \in \mathbb{N}\ [\sigma_k = f(0)f(1)f(2)\ldots f(n)]\}$; it follows that $\phi_j = f$.

These properties will now be used to define the following enumeration ψ. There are three types of indices for ψ; indices $e_{i,j,k}$ which try to produce a finite variant of the function ϕ_j on the domain W_i^η; $e_{D,0,k}$ which try to produce a finite function with domain D but might have to change the finite domain once; $e_{D,1,k}$ which produce a finite function with domain D as a second attempt after $e_{D,0,k}$ fails. Note that an index in the numbering ψ may not be chosen for all possible combinations of these parameters. The below algorithms corresponding to the parameters state explicitly when such an index is chosen and what the corresponding function in the numbering ψ does. The indices chosen are assumed to cover the natural numbers in a one-one way. All algorithms work for all k in parallel and the domain of each such function is independent of k.

– Algorithm for (i, j, k).
– Let $u_0, u_1, u_2, \ldots$ be a recursive one-one enumeration of W_i^η uniformly in i; if this set is finite then the corresponding enumeration is partial.
– Wait until $D = \{u_0, u_1, \ldots, u_{2^i \cdot 3^j - 1}\}$ is known and the corresponding elements are enumerated into W_i^η.

– Choose the index $e_{i,j,k}$; if this stage is not reached, no index for parameters (i, j, k) is chosen.
– For all $x \in D$, if $x \in \text{dom}(\sigma_k)$ then let $\psi_{e_{i,j,k}}(x) = \sigma_k(x)$ else let $\psi_{e_{i,j,k}}(x) = 0$.
– For $h = 1, 2, 3, \ldots$ do Begin
  - Let $E = \{u_\ell : 2^i 3^j 5^{h-1} \leq \ell < 2^i 3^j 5^h\}$ and wait until all elements of E are known, that is, until the first $2^i 3^j 5^h$ elements are enumerated into W_i^η.
  - Wait until $\phi_j^\eta(\ell)$ is defined on all $\ell < 2^i 3^j 5^h$.
  - For $\ell = 2^i 3^j 5^{h-1}$ to $2^i 3^j 5^h - 1$ do Begin
    If $u_\ell \in \text{dom}(\sigma_k)$ then let $\psi_{e_{i,j,k}}(u_\ell) = \sigma_k(u_\ell)$ else let $\psi_{e_{i,j,k}}(u_\ell) = \phi_j(\ell)$.
    End of for-loop for ℓ.

  End of for-loop for h.

Note that $\psi_{e_{i,j,k}}(x)$ remains undefined for all x where it is not explicitly defined in the above algorithm. The next algorithms are there to cover all functions with finite domain. The first one intends to cover the domain D but might be redirected to some other finite domain in the case that there is a domain-collision.

– Algorithm for $(D, 0, k)$.
– Choose the index $e_{D,0,k}$.
– For all $x \in D$, if $x \in \text{dom}(\sigma_k)$ then let $\psi_{e_{D,0,k}}(x) = \sigma_k(x)$ else let $\psi_{e_{D,0,k}}(x) = 0$.
– Wait until there exists in some stage s some other index d such that $W_{d,s}^\psi = D$ and there are i, j, h, D' such that either $d = e_{i,j,0} \wedge 2^i 3^j 5^h = |D|$ or $d = e_{D',0,0} \wedge |D| = 7|D'|$.
– Let E be the set of the least $6|D|$ numbers outside D.
– For all $x \in E$, if $x \in \text{dom}(\sigma_k)$ then let $\psi_{e_{D,0,k}}(x) = \sigma_k(x)$ else let $\psi_{e_{D,0,k}}(x) = 0$.
– Terminate.

In the case that D has $2^i 3^j 5^h$ elements for some i, j, h it can happen that D is temporarily equal to $W_{e_{i,j,k},s}^\psi$ but later more elements are enumerated into that set. The next case makes sure that then some other set replaces the given domain.

– Algorithm for $(D, 1, k)$.
– Determine i, j, h such that $|D| = 2^i 3^j 5^h$; if these i, j, h do not exist then abort.
– Wait for a stage s such that $W_{e_{D,0,k},s}^\psi$ has $|D| \cdot 7$ elements, index $e_{i,j,k}$ exists and $W_{e_{i,j,k},s}^\psi$ has at least $2^i 3^j 5^{h+1}$ elements.
– Choose the index $e_{D,1,k}$.
– For all $x \in D$, if $x \in \text{dom}(\sigma_k)$ then let $\psi_{e_{D,1,k}}(x) = \sigma_k(x)$ else let $\psi_{e_{D,1,k}}(x) = 0$.
– Terminate.

For the verification, it is first shown that $\text{DEQ}_\psi \leq_T \text{INF}_\eta$. This is done by showing that the following formula holds.

$W_a^\psi = W_b^\psi$ iff
either $\exists i, j, k, j', k'$ $[a = e_{i,j,k}$ and $b = e_{i,j',k'}$ and $j = j' \vee i, j, j' \in \text{INF}_\eta]$
or $\exists D, c, k, k'$ $[a = e_{D,c,k}$ and $b = e_{D,c,k'}]$.

For the correctness, note that in above constructions the parameter k does not have any influence on the domain; it only codes a finite string telling how to replace certain elements in order to get all functions covered. Therefore, it is sufficient to prove the above formula for the equivalence classes formed by considering all indices with the same parameters except for k, k' and then to take the representatives where k, k' are both 0. Now the formula is proven by case distinction.

Case $a = e_{i,j,0}$ and W_a^ψ is finite. Note that this happens if the algorithm for (i,j,k) has gone far enough to define $e_{i,j,0}$ but later gets stuck at some level h in the for-loop of the variable of the same name by waiting for sufficiently many elements to go either into W_i^ψ or into W_j^ψ to define ϕ_j. The domain has $2^i 3^j 5^{h-1}$ elements. In the case that $b = e_{i',j',0}$ then W_b^ψ is either infinite or has $2^{i'} 3^{j'} 5^{h'-1}$ elements and the domain is the same iff $i = i' \wedge j = j'$. In the case that $b = e_{D,c,0}$ then $W_b^\psi \neq W_a^\psi$: if $D = W_a^\psi$ then $\psi_{e_{D,c,0}}$ will eventually become defined on $7|D|$ elements and hence the domain is different from D while $e_{D,1,0}$ will not become created as that would require that more than $|D|$ elements go into W_a^ψ. Furthermore, no function $e_{D',c,0}$ with $D' \subset D$ has the same domain as W_a^ψ; the reason is that such functions either have the domain D' or have a domain whose cardinality is a multiple of 7.

Case $a = e_{i,j,0}$ and W_a^ψ is infinite. Note that the domain of $\psi_{e_{i,j,0}}$ is W_i^η. Then $i \in \mathrm{INF}_\eta$ and $j \in \mathrm{INF}_\eta$ as otherwise the for-loop with the variable "h" in the algorithm for (i,j,k) would get stuck with waiting for either elements to go into W_i^ψ or W_j^ψ; the latter is needed to get that ϕ_j is total. As argued in the previous case, this is the case which always applies if $i, j \in \mathrm{INF}_\eta$. Now $W_b^\psi \neq W_a^\psi$ whenever $b = e_{D,c,0}$ as a function with such an index is only defined on a finite set. Furthermore, if $b = e_{i',j',0}$ and $i \neq i'$ then W_b^ψ is either finite or equal to $W_{i'}^\eta$; in both cases $W_b^\psi \neq W_a^\psi$. The remaining case is that $b = e_{i,j',0}$ and then $W_b^\psi = W_a^\psi$ iff $W_b^\psi = W_i^\eta$ iff $j' \in \mathrm{INF}_\eta$. This verifies the formula for this case.

Case $a = e_{D,c,0}$. It follows from above case distinction that $W_a^\psi \neq W_b^\psi$ whenever b is of the form $e_{i,j,0}$. Now let i, j, h be the maximal numbers such that $2^i, 3^j, 5^h$ divide $|D|$, respectively. Consider the following two subcases.

The subcase that there is no index $e_{i,j,0}$ or there is no stage s such that $W_{e_{i,j,0},s}$ has exactly $2^i 3^j 5^h$ elements. Then for all F such that i, j, h are the maximal numbers such that $2^i, 3^j, 5^h$ divide $|F|$, respectively, satisfy that index $e_{F,1,0}$ does not exist and $W_{e_{F,0,0}}^\psi = F$. It follows that $c = 0$ and $W_b^\psi = W_a^\psi$ iff $b = e_{D,0,0}$.

The subcase that there is an index $e_{i,j,0}$ and $W_{e_{i,j,0},s}^\psi$ has at some stage s exactly $2^i 3^j 5^h$ elements. Then there is a sequence of sets $E_0, E_1, E_2, \ldots$ such that each E_n has exactly $2^i 3^j 5^h 7^n$ elements and for that n, the set $W_{e_{E_n,0,0}}^\psi$ has first the range E_n and later the range E_{n+1}. All sets $F \notin \{E_0, E_1, E_2, \ldots\}$ such that i, j, h are the maximal numbers such that $2^i, 3^j, 5^h$ divide $|F|$, respectively, satisfy that the index $e_{F,1,0}$ does not exist and $W_{e_{F,0,0}}^\psi = F$. Furthermore, the index $e_{E_0,1,0}$ exists iff $E_0 \subset W_{e_{i,j,0}}^\psi$. Now one can see the following: if $W_{e_{D,c,0}}^\psi = E_{n+1}$ for some n then $D = E_n$ and $c = 0$; if $W_{e_{D,c,0}}^\psi = E_0$ then $D = E_0$ and $c = 1$; if $W_{e_{D,c,0}}^\psi = F$ for one F as considered above in this paragraph then $D = F$ and $c = 0$. This exhausts all the possibilities for $W_{e_{D,c,0}}^\psi$. Hence $W_a^\psi = W_b^\psi$ iff $b = e_{D,c,0}$.

This case distinction completes the proof of the formula and therefore $\mathrm{DEQ}_\psi \leq_T \mathrm{INF}_\eta$.

For the converse direction, fix i as an index of $\mathbb{N}$. Note that the index $e_{i,j,0}$ exists for all j as the creation of the index does not contain any condition on j but only the condition that W_i^ψ contains at least $2^i 3^j$ elements. The mapping $j \mapsto e_{i,j,0}$ is recursive. Now $j \in \mathrm{INF}_\eta$ iff $W^\psi_{e_{i,j,0}} = W^\psi_{e_{i,i,0}}$ iff $\langle e_{i,j,0}, e_{i,i,0}\rangle \in \mathrm{DEQ}_\psi$ and hence $\mathrm{INF}_\eta \leq_m \mathrm{DEQ}_\psi$. Together with the previous result, one has $\mathrm{DEQ}_\psi \equiv_T \mathrm{INF}_\eta$.

It remains to show that the numbering ψ is universal and covers all partial recursive functions g. Given g with finite domain, let D be the domain and let k be an index of a string σ_k such that $\sigma_k(x)$ is defined and equal to $g(x)$ for all $x \in D$. There are three cases.

- There is a c with $W^\psi_{D,c,0} = D$. Then $\psi_{e_{D,c,k}}(x) = \sigma_k(x)$ for all $x \in D$ and $\psi_{e_{D,c,k}} = g$.
- $W^\psi_{e_{i,j,0}} = D$ for some i, j. Then $\psi_{e_{i,j,k}}(x) = \sigma_k(x)$ for all $x \in D$ and $\psi_{e_{i,j,k}} = g$.
- There is an F with $|D| = 7|F|$ and $W^\psi_{e_{F,0,0}} = D$. Then $\psi_{e_{F,0,k}}(x) = \sigma_k(x)$ for all $x \in D$ and $\psi_{e_{F,0,k}} = g$.

This case-distinction is exhaustive. Given g with infinite domain, there is a unique i such that W_i^η is the domain of g. Let $u_0, u_1, \ldots$ be the underlying recursive one-one enumeration of this domain considered in the construction above. There is an index j such that $\phi_j(\ell) = g(u_\ell)$ for all ℓ. Now the function $\psi_{e_{i,j,0}}$ has the domain W_i^ψ and satisfies for almost all ℓ that $\psi_{e_{i,j,0}}(u_\ell) = g(u_\ell)$. There is a k such that $\sigma_k(x)\downarrow = g(x)$ for all x in the intersection of the domains of σ_k and g and that the domain of σ_k contains all x with $\psi_{e_{i,j,0}}(x) \neq g(x)$. It follows that $\psi_{e_{i,j,k}} = g$. This completes the proof of the Theorem. ☐

Combining Proposition 22 and Theorem 23 gives the following corollary which was the main goal of these two results.

Corollary 24. *There is a universal numbering ψ such that DEQ_ψ has 1-generic Turing degree.*

4 Open Problems

In the following several major open questions of the field are identified.

Open Problem 25. *Is there a universal numbering ψ such that DMIN_ψ has minimal Turing degree?*

This is certainly possible for MIN_ψ as one can code every Turing degree below K into MIN_ψ for a suitable ψ. Recall that $\mathrm{INC}_\psi = \{\langle i,j\rangle : W_i^\psi \subseteq W_j^\psi\}$ and $\mathrm{DEQ}_\psi = \{\langle i,j\rangle : W_i^\psi = W_j^\psi\}$. Obviously

$$\mathrm{DMIN}_\psi \leq_T \mathrm{DEQ}_\psi \leq_T \mathrm{INC}_\psi \leq_T K'.$$

By Theorem 19 there is a universal numbering ψ such that $\mathrm{DMIN}_\psi <_T \mathrm{DEQ}_\psi \equiv_T K'$ and Friedberg showed that there is a domain-universal numbering ϑ for which DEQ_ϑ is recursive. Corollary 24 showed that one can make DEQ_ψ to have 1-generic Turing degree as well for some universal numbering. Hence the first two Turing reductions can be made proper while the following remains unknown.

Open Problem 26. *Is there a universal numbering* ψ *with* $\mathrm{INC}_\psi <_T K'$*?*

Note that for universal numberings, this question is equivalent to asking whether $\mathrm{INC}_\psi \not\geq_T K$. The reason is that $\mathrm{INC}_\psi \oplus K \equiv_T K'$ holds for universal numberings by $\mathrm{DMIN}_\psi \leq_T \mathrm{INC}_\psi \leq_T K'$ and Proposition 4. For domain-universal numberings, one can even ask the stronger question whether there is a domain-universal numbering ϑ with $\mathrm{INC}_\vartheta <_T K$. Kummer [10] already showed that $\mathrm{INC}_\vartheta \leq_T K$ can be obtained for some domain-universal numbering ϑ, see Theorem 13 above.

In Theorem 8 above it was shown that for numberings ψ satisfying the Kolmogorov property, $\mathrm{DMIN}^*_\psi \equiv_T K''$. On the other hand, by Theorem 19 there is a universal numbering ψ with DMIN^*_ψ being 1-generic. Although these results give already much knowledge about DMIN^*_ψ, the original problem of Schaefer [17] is still not completely solved.

Open Problem 27. *Is* $\mathrm{DMIN}^*_\psi \equiv_T K''$ *for all Gödel numberings* ψ*?*

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