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*Lowness for Weakly 1-Generic and Kurtz-Random**

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Technical Report

Foreword

Lowness for Weakly 1-Generic and Kurtz-Random[⋆]

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Abstract. It is shown that a set is low for weakly 1-generic iff it has neither dnr nor hyperimmune Turing degree. As this notion is more general than being recursively traceable, this answers negatively a recent question on the characterization of these sets. Furthermore, it is shown that every set which is low for weakly 1-generic is also low for Kurtz-random. In addition to this, it is shown that a set satisfies the notion "low for diagonally non-recursive" as introduced by Kjos-Hanssen and Nies iff it is recursive.

1 Introduction

Post [14] asked whether there is an r.e. set which lies strictly between the recursive and the complete r.e. sets. One of the several ways to settle this question is to build an r.e. set $x = \{a_0, a_1, a_2, \ldots\}$ satisfying the following two requirements:

- if e is in $W_s^{\{a_0,a_1,\ldots,a_s\}}$ for infinitely many s then $e \in W_e^x$;
- if W_e is infinite then W_e intersects x.

The first requirement induces a property called "lowness": One can compute the diagonal halting set $\{e : e \in W_e^x\}$ relative to the r.e. set x in the limit and thus the halting problem relative to x is not more complicated than the unrelativized halting problem. The second requirement is the well-known property of a set to be simple, that is, whenever the e-th r.e. set is infinite, then it intersects x. It turned out that lowness plays an important role in the theory of the recursively enumerable sets and degrees; thus a lot of work addresses questions with respect to lowness. Being such a useful tool in degree theory, it was natural that the notion of lowness was transferred to other areas like that of Algorithmic Randomness. For most notations, lowness is defined as follows.

Definition 1. Given a notation G and its relativized notation G^x. A real z is called *low for* G if for all reals y, y satisfies $\mathrm{G}^z \Leftrightarrow y$ satisfies G.

As an example, one can formalize the standard definition of low (for the jump) as follows: let r satisfy G and G^z, respectively, iff the standard halting problem and the halting problem relative

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to z, respectively, is r-recursive. Then one can see that z is low iff the halting problem relative to z has the same Turing degree as the unrelativized one. For example, a real z is low (for the jump) if for all reals r, r is above the halting problem of z iff r is above the unrelativized halting problem.

André Nies obtained one of the central results in Algorithmic Randomness by showing that the notion of H-trivial coincides with several lowness notions. The Kolmogorov complexity of a string σ is the shortest input ("program") of a fixed universal machine generating it. Changing from one universal machine to another one changes this notion only by up to a constant, therefore the choice of the universal machine itself does not matter too much. Nevertheless, one distinguishes two quite different versions of Kolmogorov complexity, the version H where the considered machines have to be prefix-free and the plain version C where no requirement on the machines are given. Here U is prefix-free if whenever $U(\sigma)$ is defined then $U(\tau)$ is undefined for all proper prefixes of σ. H-trivial reals x are now those where $H(x \upharpoonright n) \leq H(n) + c$ for some constant c and all length n where $x \upharpoonright n$ denotes the first n bits of the real $x \in \{0,1\}^\omega$. One of the applications of Kolmogorov complexity is the characterization of Martin-Löf randomness without using the tests: x is Martin-Löf random iff $H(x \upharpoonright n) \geq n$ for almost all n, which, for this paper, will now serve as an alternative definition. The characterization of Nies obtained for the H-trivial sets is now the following.

Theorem 2. (Nies [11]) *Let x be any set. Then the following statements are equivalent:*

- *x is H-trivial;*
- *x is low for H, that is, there is a constant c with $\forall \sigma\ [H(\sigma) \leq H^x(\sigma) + c]$;*
- *x is low for Martin-Löf random, that is, every Martin-Löf random set is also Martin-Löf random relative to x;*
- *x is a Δ_2^0 real which is low for Ω – here "x is low for Ω" means that the real Ω given by the halting probability $\sum_{p:\, U(p)\downarrow} 2^{-|p|}$ is Martin-Löf random relative to x.*

Nies' Theorem showed that many lowness properties connected with randomness have interesting connection with each other and also with not so related notions like H-triviality. So related notions were studied and the next theorem gives an overview of the there obtained results.

Theorem 3. *Let x be a set of natural numbers.*

1. (Nies [11]) *x is low for recursively random iff x is recursive.*
2. (Terwijn and Zambella [16]; Kjos-Hanssen, Nies and Stephan [7]) *x is low for Schnorr-random iff x is recursively traceable.*
3. (Greenberg, Miller and Yu [17]) *x is low for 1-generic iff x is recursive.*
4. (Downey, Griffiths and Reid [3]) *If x is recursively traceable than x is low for Kurtz-random; if x is low for Kurtz-random then x has hyperimmune-free Turing degree.*

Due to the last incomplete result and the fact that every weakly 1-generic set is Kurtz-random, the following two questions were asked.

Question 4 (Downey, Griffiths and Reid [3], Miller and Nies [10], Yu [17]). *Is a set x low for weakly 1-generic iff x is recursively traceable? Is x low for Kurtz-random iff x is recursively traceable?*

In the following, these two questions are refuted. Before going into the details, some background, definitions and notation will be provided.

Notation 5. The standard notation mainly follows the books of Downey and Hirschfeldt [1], Li and Vitányi [9], Odifreddi [12] and Soare [15].

In this paper, a real means an element in Cantor space $\{0,1\}^\omega$. Thus subsets of the natural numbers are identified with their characteristic function, that is, reals and subsets of natural numbers are considered to be the same. The basic open classes in Cantor space are of the form $\sigma \cdot \{0,1\}^\omega$ and have the measure $2^{-|\sigma|}$ where $|\sigma|$ is the length of σ. The measure of a class $S \subseteq \{0,1\}^\omega$ is denoted as $\mu(S)$.

Furthermore, x, y, z are used for reals, S, T, V for classes of reals, f, g, h for functions and all other lower case letters for natural numbers. As already said, C denotes the plain Kolmogorov complexity and H the prefix free Kolmogorov complexity. Strings are denoted by Greek letters σ, τ. The string $x(0)x(1)\ldots x(n)$ is denoted by $x \upharpoonright n+1$.

The two notions considered can be viewed as notions between genericity and randomness. A weakly 1-generic set is contained in every dense Σ^0_1-class and a Kurtz-random set is contained in every Σ^0_1-class of measure 1. Since such a class is always dense, it follows that every weakly 1-generic set is also Kurtz-random; but the converse does not hold as every Martin-Löf random set is Kurtz-random but many Martin-Löf random sets like Ω are not weakly 1-generic. Now the formal definition of 1-generic sets and their variants relative to a degree y is given.

Definition 6. Given reals x, y,

1. x is 1-y-generic if for every $\Sigma^0_1(y)$ class $S^y \subseteq \{0,1\}^\omega$ either $x \in S^y$ or there is an n such that $(x \upharpoonright n) \cdot \{0,1\}^\omega$ is disjoint to S^y;
2. x is weakly y-generic if $x \in S^y$ for every dense $\Sigma^0_1(y)$ class $S \subseteq \{0,1\}^\omega$;
3. x is said to be Kurtz-random relative to y if $x \in S^y$ for each $\Sigma^0_1(y)$ class S^y with $\mu(S^y) = 1$.

Note that Kurtz-random relative to the halting problem K is not the same as what Downey calls “Kurtz-2-random” as the there are Σ^0_2 classes of measure 1 which are not a $\Sigma^0_1(K)$ class.

2 Recursively Traceable and Diagonally Non-Recursive Reals

The notion of hyperimmune and hyperimmune-free sets and degrees plays from its beginning an important role in recursion theory, for the definition, see below. Already Post observed that hyperimmune sets are not hard for the halting problem with respect to truth-table reducibility [14]. On the other hand, every nonrecursive but r.e. set permits to compute a hyperimmune set. Thus they are above hyperimmune sets. From this point of view, sets which are not above hyperimmune sets are weak. Nevertheless, these sets still form a rich structure and they are well studied. The next definition summarizes the main notions. The last notion gives something like the opposite of having hyperimmune-free degree.

Definition 7.
1. Given an infinite set $x = \{n_0, n_1, n_2, \ldots\}$ with $n_0 < n_1 < n_2 <$ ..., its principal function p_x is defined by $p_x(m) = n_m$; that is, $p_x(m) = \min\{n : |x \cap \{0,1,\ldots,n\}| > m\}$. The principal functions of finite sets are partial and have a finite domain.
2. A function f majorizes an infinite set x if $\forall n\ (f(n) > p_x(n))$.

3. Given a real y, an infinite set x is y-hyperimmune if no y-computable function f majorizes x. Particularly, a set x is called hyperimmune if it is $\emptyset$-hyperimmune.
4. Given a real y, x is said to have y-hyperimmune degree if there is an infinite $z \leq_T x$ which is y-hyperimmune. Otherwise it is said that x has y-hyperimmune-free degree. In particular, x has hyperimmune-free degree if it has $\emptyset$-hyperimmune-free degree.
5. A real x is high iff there is a function $f \leq_T x$ which majorizes all infinite recursive sets y. Otherwise x is called non-high.

Given a real x of hyperimmune-free degree and a function f computable relative to x, there exists a recursive function g majorizing f. One can ask whether one can even get more information about the values computed by some function relative to x. This is not possible for all but for some reals of hyperimmune-free degree. These reals are called recursively traceable.

Definition 8. A real x is recursively traceable iff there is a recursive function h, called a bound, such that for all $f \leq_T x$ there is a recursive function g such that the $g(n)$-th canonical finite set $D_{g(n)}$ satisfies the following two properties:

- $|D_{g(n)}| \leq h(n)$;
- $f(n) \in D_{g(n)}$.

Furthermore, x is r.e. traceable if the $g(n)$-th r.e. set $W_{g(n)}$ is used instead of $D_{g(n)}$ in the definition above.

An important notion is the ability to avoid the diagonal function. As this function is partial-recursive and not total, this cannot be done with a total-recursive function. Indeed, if x has r.e. degree and avoids the diagonal function, then the degree of x is already the complete one, that is, the degree of the halting problem.

Definition 9. A real x is diagonally nonrecursive (dnr) iff there is a total function $f \leq_T x$ such that for all n either $\varphi_n(n)$ is undefined or different from $f(n)$.

Clearly, every recursively traceable x is also r.e. traceable. Indeed x is recursively traceable iff x is r.e. traceable and has hyperimmune-free degree. Note that every x of hyperimmune-free degree is non-high and generalized low$_2$. One can combine results of Kjos-Hanssen and Merkle to obtain the following theorem.

Theorem 10 (Kjos-Hanssen; Kjos-Hanssen, Merkle and Stephan [6]). *Let x be not high. Then the following are equivalent:*

1. *x is not dnr;*
2. *x is not autocomplex, that is, there is no $f \leq_T x$ such that $C(x \upharpoonright m) \geq n$ whenever $m \geq f(n)$;*
3. *for every $g \leq_T x$ there is a recursive function h such that $g(n) = h(n)$ infinitely often;*
4. *x is infinitely often traceable in the sense that there is a recursive function h such that for all $f \leq_T x$ there is a recursive function g with $\forall n\,(|D_{g(n)}| \leq h(n))$ and $\exists^\infty n\,(f(n) \in D_{g(n)})$;*
5. *for every unbounded and nondecreasing recursive function h and every function $g \leq_T x$ there are infinitely many n with $C(g(n)) < h(n)$.*

Furthermore, if the Turing degree of x is neither hyperimmune nor dnr, then one can strengthen the third point as follows: for every $g \leq_T x$ there are recursive functions $\tilde{h}, h$ such that

$$\forall n\ \exists m \in \{n, n+1, \ldots, \tilde{h}(n)\}\ (h(m) = g(m)).$$

Autocomplex sets are not r.e. traceable and vice versa. But these notions do also not partition the class of all reals; the next result shows that there is a whole Π^0_1 class containing reals which are neither r.e. traceable nor autocomplex. This result covers the well-known examples of reals which are neither r.e. traceable nor autocomplex: (a) as the r.e. Turing degrees form a basis for Π^0_1-classes [12, Exercise V.5.33, volume 1, pages 508 and 509], there is an x of r.e. degree which is neither r.e. traceable nor autocomplex; (b) by Jockusch and Soare's Hyperimmune-Free Basis Theorem [12, Proposition V.5.34, volume 1, page 509], there is an x of hyperimmune-free degree which is neither r.e. traceable nor autocomplex. Result (a) is quite direct as every r.e. set which is neither Turing complete nor low$_2$ has this property. Result (b) can be obtained by considering sets which are generic for "very strong array forcing" as considered by Downey, Jockusch and Stob [2, 7]; Kjos-Hanssen pointed out to the authors that those sets are neither autocomplex nor r.e. traceable nor do they have hyperimmune Turing degree. An application of the following result would be that there are reals which are low for Ω but neither recursively traceable nor dnr. This application can be obtained via a theorem of Downey, Hirschfeldt, Miller and Nies [4] which says that every nonempty Π^0_1-class has a member which is low for Ω.

Proposition 11. *There is a partial-recursive $\{0,1\}$-valued function ψ with coinfinite domain such that every x extending ψ is neither autocomplex nor r.e. traceable.*

Proof. The function ψ is constructed such that

1. $\psi(2^n)$ is undefined for infinitely many n;
2. if $\psi(2^n)$ is undefined and x is a total extension of ψ and $m \geq 2^{n+1}$ then $C(x \upharpoonright m) \geq n-1$;
3. if $\psi(2^n)$ is undefined, x is a total extension of ψ and $\varphi^x_e(3^n)$ terminates such that the maximum of its computation-time, largest query and computation-result is s for some $e \leq n$ and $s \geq 2n$ then $\psi(m)$ is defined for $m = 2^n, 2^n+1, \ldots, 2^s-1$.

Now these three conditions are used to show that a given total extension x of ψ is neither r.e. traceable nor autocomplex.

Assume a recursive bound h is given. Let $f(m) = C(x \upharpoonright 2^{h(m+1)+m+4})$. Choose m, n such that $h(m)+m+3 \leq n < h(m+1)+m+4$ and $\psi(n)$ is undefined. There are infinitely many m for which there is such an n by the first condition above. Now $C(f(m)) > n$ and $C(f(m) \mid m) > h(m)$, for infinitely many m, thus x is not r.e. traceable with bound h. So x is not r.e. traceable at all.

Furthermore, if φ^x_e is total and $n > e$ then $\varphi^x_e(3^n)$ queries x at places m where either $\psi(m)$ is defined or $m < 2^{n+1}$. Therefore, one can compute $\varphi^x_e(3^n)$ and $x \upharpoonright \varphi^x_e(3^n)$ from n and $x \upharpoonright 2^{n+1}$, thus $C(x \upharpoonright \varphi^x_e(3^n)) < 3^n$. It follows that φ^x_e does not witness that x is autocomplex. So x is neither autocomplex nor dnr.

It remains to show that the considered ψ really exists. Let U be a universal machine for the complexity C and for a string τ in the domain of U, let $\mathrm{bv}(\tau)$ be the value of binary number 1τ. Now one constructs ψ in stages as follows. ψ_0 is everywhere undefined and in stage $s+1$ the following is done.

1. Begin Stage $s+1$.
2. Find the smallest n for which there are e, m, x, t such that $e \leq n$, $2^{n+1} \leq m \leq t \leq s$, x extends ψ_s, $\psi_s(2^n)\uparrow$, $\psi_s(m)\uparrow$ and $\varphi^x_e(3^n)$ terminates such that the maximum of its computation-time, largest query and computation-result is exactly t.
3. If n with e, m, x, t are found in Step 2 then let, for all $k \in \{2^n, 2^n+1, \ldots, 2^{s+1}-1\}$ where $\psi_s(k)$ is undefined, $\psi_{s+1}(k) = x(k)$.

4. For all $\tau \in \{0,1\}^*$ and i, j such that $\mathrm{bv}(\tau) < 2^i < 2^s$, $U_s(\tau)\downarrow$, $j = 2^i + \mathrm{bv}(\tau) < |U_s(\tau)|$ and $\psi_s(j)\uparrow$, let $\psi_{s+1}(j) = 1 - U_s(\tau)(j)$.
5. End Stage $s+1$.

Note that $\psi(2^n)$ can only become defined by activities in Step 3 of some stage. One can show by the usual finite injury arguments that there are infinitely many n for which $\psi(2^n)$ remains undefined. Furthermore, whenever $\psi(2^n)$ is undefined and $|\tau| < n-1$ then $j = 2^n + \mathrm{bv}(\tau)$ satisfies that either $U(\tau)$ is undefined or $U(\tau)(j), \psi(j)$ are both undefined or $U(\tau)(j), \psi(j)$ are both defined and different where, for the string $U(\tau)$, $U(\tau)(j)$ is the bit at position $j+1$ if the length is at least $j+1$ and is undefined if the length is at most j. As the mapping $\tau, n \rightarrow 2^n + \mathrm{bv}(\tau)$ is one-one on the domain of all τ, n with $\mathrm{bv}(\tau) < 2^n$ and as Step 3, for every n, makes either ψ either on a whole interval $\{2^n, 2^n+1, \ldots, 2^{n+1}-1\}$ or does not change ψ on the interval at all, it follows that if $\psi(2^n)$ is undefined then Step 4 guarantees that $C(x \upharpoonright m) \geq n-1$ for all $m \geq 2^{n+1}$. Furthermore, compactness ensures that after finitely many stages, Step 3 of the construction has ensured that the third condition on ψ is also satisfied. This completes the verification of the construction of ψ. $\blacksquare$

3 Lowness for Weakly 1-Generic

The next result characterizes when a set is low for weakly 1-generic.

Theorem 12. *The following statements are equivalent for every real x,*

1. *Every dense $\Sigma^0_1(x)$ class $S^x \subseteq \{0,1\}^\omega$ has a dense Σ^0_1 subclass.*
2. *x is low for weakly 1-generic.*
3. *The degree of x is hyperimmune-free and each 1-generic real is weakly 1-x-generic.*
4. *The degree of x is hyperimmune-free and not dnr.*

Proof. Obviously, the first statement implies the second. Kurtz [8] showed that every hyperimmune degree contains a weakly 1-generic real and thus the second statement implies the third. Proposition 13 below proves that the third statement implies the fourth. The implication from the fourth to the first condition follows from Theorem 14 below. $\blacksquare$

Proposition 13. *If each 1-generic real is weakly 1-x-generic, then x is not dnr.*

Proof. Assume by way of contradiction that x is dnr and every 1-generic set y is also weakly 1-x-generic. Nies [11] showed that there exists a 1-generic and H-trivial real y. Furthermore, as x is dnr, x is autocomplex [6]. So there is an x-recursive function f such that $H(x \upharpoonright m) \geq n$ for all $m \geq f(n)$. Without loss of generality, $f(n)$ queries x only below $f(n)$ when computing this value; otherwise one could replace $f(n)$ by the maximum of $f(n)$ and all places queried during the computation. Now one defines S as

$$S = \{\sigma(x \upharpoonright f(|\sigma|)) : \sigma \in \{0,1\}^*\}$$

and observes that S is dense. By assumption, y is weakly x-generic. So there are infinitely many n such that $(y \upharpoonright n)(x \upharpoonright f(n)) \preceq y$. Given such an n, one can compute $f(n)$ relative to y by querying $y(m+n)$ whenever the original computation of f queries $x(m)$, the reason is that whenever $x(m)$ is queried in this computation, then $m < f(n)$ and $y(m+n) = x(m)$. Using the property that H^y and H differ by up to a constant for y as y is H-trivial [11] as well as the fact that H is autocomplex, one has that

$$\begin{array}{rcccl} H^y(x \upharpoonright f(n)) & \leq & H^y(n, y \upharpoonright n + f(n)) + c_1 & \leq & H^y(n, f(n)) + c_2 \\ & \leq & H^y(n) + c_3 & \leq & H(n) + c_4 \end{array}$$

for some constants c_1, c_2, c_3, c_4 and the infinitely many n with $(y \upharpoonright n)(x \upharpoonright f(n)) \preceq y$. It follows that $H(n) \geq n - c_4$ for infinitely many n, a contradiction. ■

4 Low for Kurtz-Random

Downey, Griffiths and Reid [3] conjectured that every low for Kurtz-random real is recursively traceable. The following theorem refutes the conjecture.

Theorem 14. *Let x have neither hyperimmune nor dnr Turing degree. Then the following two statements hold.*

1. *Every $\Sigma^0_1(x)$ class S^x of measure 1 has a Σ^0_1 subclass T of measure 1.*
2. *Every dense $\Sigma^0_1(x)$ class has a dense Σ^0_1 subclass.*

In particular, x is low for Kurtz-random and low for weakly 1-generic.

Proof. If S^x has measure 1 then S^x is dense: otherwise there would be a σ such that $\sigma \cdot \{0,1\}^\omega$ is disjoint to S^x and $\mu(S^x) \leq 1 - 2^{-|\sigma|}$. The proof is now given for the first statement where S^x has measure 1 and is dense. The proof for the second statement where S^x is only dense can be obtained from this proof by just omitting all conditions and constraints dealing with the measure of classes.

The argument in the proof is somewhat similar to the one in [17]. But the proof is greatly simplified due to Theorem 10. Fix x such that the Turing degree of x is neither hyperimmune nor dnr and consider any dense Σ^0_1 class S^x. For S^x, there is a function $\hat{f} \leq_T x$ such that, for all n,

- $\hat{f}(n) > n$;
- $\forall \sigma \in \{0,1\}^n \exists \tau \in \{0,1\}^{\hat{f}(n)}\ (\sigma \preceq \tau \wedge \tau \cdot \{0,1\}^\omega \subseteq S^x)$;
- $\mu(\{y \in S^x : (y \upharpoonright \hat{f}(n)) \cdot \{0,1\}^\omega \subseteq S^x\}) \geq 1 - 2^{-n}$.

Since x has hyperimmune-free Turing degree, there is a recursive function f such that, for all n, $f(n+1) > \hat{f}(f(n))$. Then there is an x-recursive function g such that, for all n,

- $g(n) \subseteq \{0,1\}^{f(n+1)}$;
- $\forall \sigma \in \{0,1\}^{f(n)} \exists \tau \in g(n)\ (\sigma \preceq \tau)$;
- $\mu(g(n) \cdot \{0,1\}^\omega) \geq 1 - 2^{-n}$;
- $g(n) \cdot \{0,1\}^\omega \subseteq S^x$.

As the Turing degree of x is neither dnr nor hyperimmune, there are recursive functions $h, \tilde{h}$ such that, for all n,

- $h(n) \subseteq \{0,1\}^{f(n+1)}$;
- $\forall \sigma \in \{0,1\}^{f(n)} \exists \tau \in h(n)\ (\sigma \preceq \tau)$;
- $\mu(h(n) \cdot \{0,1\}^\omega) \geq 1 - 2^{-n}$;

- $\exists m \in \{n, n+1, \ldots, \tilde{h}(n)\}\ (h(m) = g(m))$.

Now one can define the Σ^0_1 class T as

$$T = \{x : \exists n \forall m \in \{n, n+1, \ldots, \tilde{h}(n)\}\ (x \upharpoonright f(m+1) \in h(m))\}.$$

The class T is dense because for every $\sigma \in \{0,1\}^n$ there is a $\tau_{n-1} \in \{0,1\}^{f(n)}$ extending σ as $f(n) > n$ and a sequence of $\tau_m \in h(m)$ each extending τ_{m-1} for $m = n, n+1, \ldots, \tilde{h}(n)$. Then $\tau_{\tilde{h}(n)} \cdot \{0,1\}^\omega \subseteq T$. The measure of T is 1 as

$$\mu(\{x : \forall m \in \{n, n+1, \ldots, \tilde{h}(n)\}\ (x \upharpoonright m \in h(m))\}) \geq 1 - 2^{-n-1}$$

for all n. Furthermore, for every $x \in T$ there is an n and $m \in \{n, n+1, \ldots, \tilde{h}(n)\}$ such that $h(m) = g(m)$ and $x \in uh(m)$. Thus $x \in g(m) \cdot \{0,1\}^\omega$ and $T \subseteq S^x$. ■

It is open whether lowness for Kurtz-random is equivalent to lowness for weakly 1-generic.

5 A Note on Lowness for DNR

Say that a function f avoids the x-diagonal function iff $f(n) \neq \varphi^x_n(n)$ whenever $\varphi^x_n(n)$ is defined; if $x = \emptyset$ then f avoids the diagonal function. Kjos-Hanssen was interested in the question when a set is low for dnr and considered the following two notions:

- x is low for dnr in the first sense iff for all y, some $f \leq_T y$ avoids the diagonal function iff some $g \leq_T x \oplus y$ avoids the x-diagonal function;
- x is low for dnr in the second sense iff for all y, some $f \leq_T y$ avoids the diagonal function iff some $g \leq_T y$ avoids the x-diagonal function.

In the definition of low for dnr in the first sense, all computations are properly relativized. For that reason, after some discussions, Nies and Kjos-Hanssen agreed to state it in the survey of Nies and Miller [10, Question 5.1] as an open question whether there are nonrecursive sets which are low for dnr in the first sense but to ignore the second notion.

The answer to this question is "no": So given any nonrecursive x, Posner and Robinson [13] showed that there is a 1-generic y such that $x' \leq_T x \oplus y$. Then y is dnr relative to x since y can compute x' relative to x. On the other hand, y is not dnr since y is 1-generic. This shows that only the recursive sets are low for dnr in the first sense.

So it remains to ask what can be said about the low for dnr in the second sense and to characterize these sets. Figueira, Nies and Stephan [5] called a set x jump-traceable iff there is a recursive function h such that $H(\varphi^x_n(n)) \leq h(n)$ for all n where $\varphi^x_n(n)$ is defined. They showed that there are uncountably many jump-traceable sets. Fix now a jump-traceable x and the corresponding recursive function h. If y is dnr then there is a function $g \leq_T y$ with $H(g(n)) > h(n)$ for all n. This function g satisfies $g(n) \neq \varphi^x_n(n)$ for all n where the latter is defined. Hence, low for dnr in the second sense is a non-trivial property satisfied by all jump-traceable sets and is worth further studies. Furthermore, all the x which are low for dnr in the second sense are positive examples for the following class of degrees in which Kjos-Hanssen is interested as well.

- Which x satisfy for all y that whenever some $f \leq_T y$ avoids the diagonal function then some $g \leq_T x \oplus y$ avoids the x-diagonal function?

Nevertheless, it remains open whether there are further x which satisfy this question but are not low for dnr in the second sense.

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References

1. Rodney G. Downey and Denis Hirschfeldt. *Algorithmic randomness and complexity.* Springer, Heidelberg. To appear.
2. Rod Downey, Carl G. Jockusch and Michael Stob. Array nonrecursive degrees and genericity. *Computability, Enumerability, Unsolvability. London Mathematical Society Lecture Notes Series*, 224:93–104, 1996.
3. Rodney G. Downey, Evan J. Griffiths and Stephanie Reid. On Kurtz randomness. *Theoretical Computer Science*, 321(2-3):249–270, 2004.
4. Rod Downey, Denis Hirschfeldt, Joseph Miller and André Nies, Relativizing Chaitin's halting probability, *Journal of Mathematical Logic* 5:167–192, 2005.
5. Santiago Figueira, André Nies and Frank Stephan. Lowness properties and approximations of the jump. *Proceedings of the Twelfth Workshop of Logic, Language, Information and Computation* (WoLLIC 2005). Electronic Lecture Notes in Theoretical Computer Science 143:45–57, 2006.
6. Bjørn Kjos-Hanssen, Wolfgang Merkle and Frank Stephan. Kolmogorov Complexity and the Recursion Theorem. *Twentythird Annual Symposium on Theoretical Aspects of Computer Science*, Marseille, France, February 23-25, 2006, Proceedings. *Lecture Notes in Computer Science*, 3884:149–161, 2006.
7. Bjørn Kjos-Hanssen, André Nies and Frank Stephan. Lowness for the class of Schnorr random reals. *SIAM Journal on Computing.* To appear.
8. Stuart Kurtz. *Randomness and genericity in the degrees of unsolvability.* Ph.D. Dissertation, University of Illinois, Urbana, 1981.
9. Ming Li and Paul Vitányi. *An introduction to Kolmogorov complexity and its applications.* Texts and Monographs in Computer Science. Springer, New York, 1993.
10. Joseph S. Miller and André Nies. Randomness and computability: Open questions. *The Bulletin of Symbolic Logic.* To appear.
11. André Nies. Lowness properties and randomness. *Advances in Mathematics*, 197(1):274–305, 2005.
12. Piergiorgio Odifreddi. *Classical Recursion Theory.* North-Holland, Amsterdam, 1989.
13. David B. Posner and Robert W. Robinson. Degrees joining to $\mathbf{0}'$. *The Journal of Symbolic Logic* 46:714–722, 1981.
14. Emil Post. Recursively enumerable sets of positive integers and their decision problems, *Bulletin of the American Mathematical Society*, 50:284–316, 1944.
15. Robert I. Soare. *Recursively enumerable sets and degrees.* Springer, Heidelberg, 1987.
16. Sebastiaan A. Terwijn and Domenico Zambella. Computational randomness and lowness. *The Journal of Symbolic Logic*, 66(3):1199–1205, 2001.
17. Liang Yu. Lowness for genericity. *Archive for Mathematical Logic*, 45(2):233–238, 2006.