On intrinsic complexity of learning geometrical concepts from texts

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1 Introduction

The goal of this paper is to quantify complexity of algorithmic learning geometrical concepts from growing finite segments. Consider, for example, an *open semi-hull* representing the space consisting of all points (x, y) with integer components x, y in the first quadrant of the plane bounded by the y-axis and the "broken" line passing through some points $(a_0, c_0), (a_1, c_1), \ldots, (a_n, c_n), a_i, c_i \in N, a_i < a_{i+1}, 0 \leq i < n$. The line is straight between any points $(a_i, c_i), (a_{i+1}, c_{i+1}))$ and begins at $(a_0, c_0) = (0, 0)$; further we assume that the slope of the broken line is monotonically non-decreasing — that is $(c_{i+1} - c_i)/(a_{i+1} - a_i)$ is non-decreasing in i (we need this for learnability of the semi-hull). For technical ease we further assume that the first line segment $(0, 0), (a_1, c_1)$ is adjacent to the x-axis, that is, $c_1 = 0$. Any such open semi-hull can be easily learned in the limit by the following strategy: given growing finite sets of points in the open hull (potentially getting all points), learn the first "break" point $(a_1, c_1)(c_1 = 0)$, then the first slope $(c_2 - c_1)/(a_2 - a_1)$, then the second "break" point (a_2, c_2), then the second slope $(c_3 - c_2)/(a_3 - a_2)$, etc. Is this strategy optimal? Do there exist "easier" strategies of learning such geometrical concepts? How to measure complexity of learning such concepts? The theory of inductive inference (learning from growing finite segments) suggests several ways to quantify complexity of learning. Among them are:

a) counting the number of *mind changes* [BF72, CS83, LZ93] the learner makes before arriving to the correct hypothesis;

b) measuring the amount of so-called *long-term* memory the learner uses [Kin94, KS95];

c) reductions between different learning problems (classes of languages) and respective degrees of so-called *intrinsic* complexity [FKS95, JS96, JS97].

There have been other notions of complexity of learning considered in literature (for example see [Gol67, DS86, Wie86]).

The first two approaches, however, cannot capture the complexity of learning open semi-hulls with different numbers of "angles": the number of mind changes cannot be bounded by any constant, and the long-term memory is maximum (linear) even for one "angle". Thus, we have chosen reductions as the way to measure (relative) complexity of geometrical concepts like open semi-hulls. An important issue here is which classes of languages can be used as a "scale" for quantifying complexity of open semi-hulls. One such "scale", that turned out to be appropriate for our goal, had been suggested in [JKW99]. This "scale" is a hierarchy of degrees of intrinsic complexity composed of "simple natural ground" degrees. Every such "natural ground" degree represents a natural type of learning strategy. For example, the degree $INIT$ represents a strategy that tries to use a sequence of hypotheses "equivalent" to a sequence of monotonically growing finite sets. Another such "ground" degree, $COINIT$, tries to use a sequence of hypotheses "equivalent" to a sequence of monotonically growing sets $N_a = \{x | x \in N, x \geq a\}$. Intuitively, capabilities of $INIT$ and $COINIT$-strategies must be different, and it has been formally established in [JS96]. It has been also demonstrated in [JS96] that many important "simple" learning problems (in particular, *pattern languages*) can be handled by strategies of these "basic" types. Now, the corresponding degrees of complexity $INIT$ and $COINIT$ can be used to form a complex hierarchy of degrees/strategies as follows. Imagine a "three-dimensional" language L. Suppose an $INIT$-type strategy M_1 can be used to learn its first "dimension", L_1. Once this "dimension" has been learned, a strategy of a different (or even same) type, say, $COINIT$ can pick the "grammar" learned by M_1 and use this information to learn the second "dimension" L_2. Consequently, the grammar learned by the $COINIT$-strategy M_2 can be used to learn the third "dimension" L_3 by a strategy M_3 of type, say, $INIT$. Thus, we get a strategy of the type $(INIT, COINIT, INIT)$, where information learned by the learner M_i is relayed to the learner M_{i+1} making it possible to learn the next "dimension". This idea can be naturally extended to any (finite) number of "dimensions" and to any sequences $Q = (q_1, q_1, ..., q_k)$ of strategies $q_i \in \{INIT, COINIT\}$. It has been shown in [JKW99] that the degrees of complexity (classes of languages) corresponding to such Q-strategies form a rich hierarchy. For example, some classes learnable by $(INIT, COINIT, INIT)$-type strategies cannot be learned by any $(INIT, COINIT)$-strategy. Placing a learning problem $SEMI\_HULL$ (representing the above open semi-hulls) into this hierarchy, that is, finding a Q such that $SEMI\_HULL \equiv Q$, or finding Q_1, Q_2 such that $SEMI\_HULL$ is "between" Q_1, Q_2 (where Q_1, Q_2 would be as "close" as possible), would determine what is the most "efficient" learning strategy for open semi-hulls of the above type, and which strategies cannot learn them. The reader may have noticed that learning every "break" point and every slope requires a "simple" strategy of certain type, and they can be learned only in certain order, with information learned on any step being used by the next learning strategy. Thus, we hope it will not be surprising that open semi-hulls will fit well into the above Q-hierarchy. What will be surprising, though, is the fact that the natural learning strategy for open semi-hulls, as described in the first paragraph, is not "optimal".

The paper has the following structure. Section 2 introduces notation. In Section 3 we define reductions and degrees of complexity. In Section 4 we give formal definition of Q-classes and degrees. This definition extends the definition of Q-classes in [JKW99]: in addition to classes $INIT$ and $COINIT$, we use in vectors Q a new class of strategies/languages $HALF$ that turns out to be different from $INIT$ and $COINIT$ and useful for classifying geometrical concepts. In Section 5 we show that Q-hierarchy can be appropriately extended to class of learning strategies/languages $HALF$. In Section 6 we define our main concept, the classes of the type $SEMI\_HULL$ that formalize intuitive geometrical concept described above; in this section we also prove some useful technical propositions. In Sections 7 and 8 we establish upper and lower bounds for the $SEMI\_HULL$ degrees in terms of Q-hierarchy. We also show that although the bounds do not match, no better lower or upper bounds can be achieved. In Section 9 we introduce the classes $coSEMI\_HULL$ that consist of complements of languages in $SEMI\_HULL$. Sections 10 and 11 establish lower and upper bounds for $coSEMI_HULL$s in terms of Q-hierarchy.

Upper and lower bounds for $SEMI_HULL$s and $coSEMI_HULL$s come close, but do not match (though, upper bounds are much "lower" than the ones suggested by "intuitive" strategies learning classes in question). In Section 12 we define classes of *open hulls* formed by intersections of languages in *SEMI_HULL*s adjacent to x and y-axis; for these classes we have established matching upper and lower bounds on complexity of their learning in terms of Q-hierarchy. In Section 13 we define the classes of languages formed by complements of open hulls and establish matching upper and lower bounds for the corresponding degrees of intrinsic complexity. All the abovementioned upper bounds are much "lower" that the ones suggested by "intuitive" learning strategies.

2 Notation and Preliminaries

Any unexplained recursion theoretic notation is from [Rog67]. The symbol N denotes the set of natural numbers, $\{0, 1, 2, 3, \ldots\}$. Z denotes the set of integers. Z^- denotes the set of negative integers. Symbols $\emptyset$, $\subseteq$, $\subset$, $\supseteq$, and $\supset$ denote empty set, subset, proper subset, superset, and proper superset, respectively. $D_0, D_1, \ldots$, denotes a canonical recursive indexing of all the finite sets [Rog67]. We assume that if $D_i \subseteq D_j$ then $i \leq j$ (the canonical indexing defined in [Rog67] satisfies this property). Cardinality of a set S is denoted by $\mathrm{card}(S)$. The maximum and minimum of a set are denoted by $\max(\cdot), \min(\cdot)$, respectively, where $\max(\emptyset) = 0$ and $\min(\emptyset) = \infty$.

We let $\langle \cdot, \cdot \rangle$ stand for an arbitrary, computable, bijective mapping from $N \times N$ onto N [Rog67]. We assume without loss of generality that $\langle \cdot, \cdot \rangle$ is monotonically increasing in both its arguments. We define $\pi_1(\langle x, y \rangle) = x$ and $\pi_2(\langle x, y \rangle) = y$. $\langle \cdot, \cdot \rangle$ can be extended to n-tuples in a natural way (including $n = 1$, where $\langle x \rangle$ may be taken to be x). Projection functions $\pi_1, \ldots, \pi_n$ corresponding to n-tuples can be defined similarly (where the tuple size would be clear from context). Due to above isomorphism between N^k and N, we often identify $(x_1, \cdots, x_n)$ with $\langle x_1, \cdots, x_n \rangle$. Thus we can say $L_1 \times L_2 = \{\langle x, y \rangle \mid x \in L_1, y \in L_2\}$.

By φ we denote a fixed *acceptable* programming system for the partial computable functions: $N \rightarrow N$ [Rog67, MY78]. By φ_i we denote the partial computable function computed by the program with number i in the φ-system. Symbol $\mathcal{R}$ denotes the set of all total computable functions. By Φ we denote an arbitrary fixed Blum complexity measure [Blu67, HU79] for the φ-system. By W_i we denote $\mathrm{domain}(\varphi_i)$. W_i is, then, the r.e. set/language ($\subseteq N$) accepted (or equivalently, generated) by the φ-program i. We also say that i is a grammar for W_i. Symbol $\mathcal{E}$ will denote the set of all r.e. languages. Symbol L, with or without decorations, ranges over $\mathcal{E}$. By $\overline{L}$, we denote the complement of L, that is $N - L$. Symbol $\mathcal{L}$, with or without decorations, ranges over subsets of $\mathcal{E}$. We denote by $W_{i,s}$ the set $\{x \leq s \mid \Phi_i(x) < s\}$.

$\downarrow$ denotes defined or converges. $\uparrow$ denotes undefined or diverges.

A partial function F from N to N is said to be partial limit recursive, iff there exists a recursive function f from $N \times N$ to N such that for all x, $F(x) = \lim_{y \rightarrow \infty} f(x, y)$. Here if $F(x)$ is not defined then $\lim_{y \rightarrow \infty} f(x, y)$, must also be undefined. A partial limit recursive function F is called (total) limit recursive function, if F is total.

We now present concepts from language learning theory. The next definition introduces the concept of a *sequence* of data.

Definition 1 (a) A *finite sequence* σ is a mapping from an initial segment of N into $(N \cup \{\#\})$. The empty sequence is denoted by Λ.

(b) The *content* of a finite sequence σ, denoted $\mathrm{content}(\sigma)$, is the set of natural numbers in the range of σ.

(c) The *length* of σ, denoted by $|\sigma|$, is the number of elements in σ. So, $|\Lambda| = 0$.

(d) For $n \leq |\sigma|$, the initial sequence of σ of length n is denoted by $\sigma[n]$. So, $\sigma[0]$ is Λ.

Intuitively, $\#$'s represent pauses in the presentation of data. We let σ, τ, and γ, with or without decorations, range over finite sequences. We denote the sequence formed by the concatenation of τ at the end of σ by $\sigma \diamond \tau$. Sometimes we abuse the notation and use $\sigma \diamond x$ to denote the concatenation of sequence σ and the sequence of length 1 which contains the element x. SEQ denotes the set of all finite sequences.

Definition 2 [Gol67] (a) A *text* T for a language L is a mapping from N into $(N \cup \{\#\})$ such that L is the set of natural numbers in the range of T.

(b) The *content* of a text T, denoted content(T), is the set of natural numbers in the range of T.

(c) $T[n]$ denotes the finite initial sequence of T with length n.

We let T, with or without decorations, range over texts. We let $\mathcal{T}$ range over sets of texts.

Definition 3 [Gol67] A *language learning machine* is an algorithmic device which computes a mapping from SEQ into N.

We let $\mathbf{M}$, with or without decorations, range over learning machines. $\mathbf{M}(T[n])$ is interpreted as the grammar (index for an accepting program) conjectured by the learning machine $\mathbf{M}$ on the initial sequence $T[n]$. We say that $\mathbf{M}$ converges on T to i, (written $\mathbf{M}(T)\downarrow = i$) if $(\stackrel{\infty}{\forall} n)[\mathbf{M}(T[n]) = i]$.

There are several criteria for a learning machine to be successful on a language. Below we define identification in the limit introduced by Gold [Gol67].

Definition 4 [Gol67, CS83] Suppose $a \in N \cup \{*\}$.

(a) $\mathbf{M}$ $\mathbf{TxtEx}$-identifies a text T just in case $(\exists i \mid W_i = \text{content}(T))$ $(\stackrel{\infty}{\forall} n)[\mathbf{M}(T[n]) = i]$.

(b) $\mathbf{M}$ $\mathbf{TxtEx}$-identifies an r.e. language L (written: $L \in \mathbf{TxtEx}(\mathbf{M})$) just in case $\mathbf{M}$ $\mathbf{TxtEx}$-identifies each text for L.

(c) $\mathbf{M}$ $\mathbf{TxtEx}$-identifies a class $\mathcal{L}$ of r.e. languages (written: $\mathcal{L} \subseteq \mathbf{TxtEx}(\mathbf{M})$) iff $\mathbf{M}$ $\mathbf{TxtEx}$-identifies each $L \in \mathcal{L}$.

(d) $\mathbf{TxtEx} = \{\mathcal{L} \subseteq \mathcal{E} \mid (\exists \mathbf{M})[\mathcal{L} \subseteq \mathbf{TxtEx}(\mathbf{M})]\}$.

Other criteria of success are finite identification [Gol67], behaviorally correct identification [Fel72, OW82, CL82], and vacillatory identification [OW82, Cas88]. In the present paper, we only discuss results about $\mathbf{TxtEx}$-identification.

3 Reductions

We first present some technical machinery.

We write $\sigma \subseteq \tau$ if σ is an initial segment of τ, and $\sigma \subset \tau$ if σ is a proper initial segment of τ. Likewise, we write $\sigma \subset T$ if σ is an initial finite sequence of text T. Let finite sequences $\sigma^0, \sigma^1, \sigma^2, \ldots$ be given such that $\sigma^0 \subseteq \sigma^1 \subseteq \sigma^2 \subseteq \cdots$ and $\lim_{i\to\infty} |\sigma^i| = \infty$. Then there is a unique text T such that for all $n \in N$, $\sigma^n = T[|\sigma^n|]$. This text is denoted by $\bigcup_n \sigma^n$. Let $\mathbf{T}$ denote the set of all texts, that is, the set of all infinite sequences over $N \cup \{\#\}$.

We define an *enumeration operator* (or just operator), Θ, to be an algorithmic mapping from SEQ into SEQ such that for all $\sigma, \tau \in$ SEQ, if $\sigma \subseteq \tau$, then $\Theta(\sigma) \subseteq \Theta(\tau)$. We further assume that for all texts T, $\lim_{n\to\infty} |\Theta(T[n])| = \infty$. By extension, we think of Θ as also defining a mapping from $\mathbf{T}$ into $\mathbf{T}$ such that $\Theta(T) = \bigcup_n \Theta(T[n])$.

A final notation about the operator Θ. If for a language L, there exists an L' such that for each text T for L, $\Theta(T)$ is a text for L', then we write $\Theta(L) = L'$, else we say that $\Theta(L)$

is undefined. The reader should note the overloading of this notation because the type of the argument to Θ could be a sequence, a text, or a language; it will be clear from the context which usage is intended.

We let $\Theta(\mathcal{T}) = \{\Theta(T) \mid T \in \mathcal{T}\}$, and $\Theta(\mathcal{L}) = \{\Theta(L) \mid L \in \mathcal{L}\}$.

We also need the notion of an infinite sequence of grammars. We let α, with or without decorations, range over infinite sequences of grammars. From the discussion in the previous section it is clear that infinite sequences of grammars are essentially infinite sequences over N. Hence, we adopt the machinery defined for sequences and texts over to finite sequences of grammars and infinite sequences of grammars. So, if $\alpha = i_0, i_1, i_2, i_3, \ldots$, then $\alpha[3]$ denotes the sequence i_0, i_1, i_2, and $\alpha(3)$ is i_3. Furthermore, we say that α converges to i if there exists an n such that, for all $n' \geq n$, $i_{n'} = i$.

Let $\mathbf{I}$ be any criterion for language identification from texts, for example $\mathbf{I} = \mathbf{TxtEx}$. We say that an infinite sequence α of grammars is $\mathbf{I}$-*admissible* for text T just in case α witnesses $\mathbf{I}$-identification of text T. So, if $\alpha = i_0,\ i_1,\ i_2, \ldots$ is a $\mathbf{TxtEx}$-admissible sequence for T, then α converges to some i such that $W_i = \text{content}(T)$; that is, the limit i of the sequence α is a grammar for the language content(T).

We now formally introduce our reductions. Although in this paper we will only be concerned with $\mathbf{TxtEx}$-identification, we present the general case of the definition.

Definition 5 [JS96] Let $\mathcal{L}_1 \subseteq \mathcal{E}$ and $\mathcal{L}_2 \subseteq \mathcal{E}$ be given. Let identification criterion $\mathbf{I}$ be given. Let $\mathcal{T}_1 = \{T \mid T \text{ is a text for } L \in \mathcal{L}_1\}$. Let $\mathcal{T}_2 = \{T \mid T \text{ is a text for } L \in \mathcal{L}_2\}$. We say that $\mathcal{L}_1 \leq^{\mathbf{I}} \mathcal{L}_2$ just in case there exist operators Θ and Ψ such that for all $T, T' \in \mathcal{T}_1$ and for all infinite sequences α of grammars the following hold:

(a) $\Theta(T) \in \mathcal{T}_2$ and

(b) if α is an $\mathbf{I}$-admissible sequence for $\Theta(T)$, then $\Psi(\alpha)$ is an $\mathbf{I}$-admissible sequence for T.

(c) if content$(T) =$ content(T') then content$(\Theta(T)) =$ content$(\Theta(T'))$.

We say that $\mathcal{L}_1 \equiv^{\mathbf{I}} \mathcal{L}_2$ iff $\mathcal{L}_1 \leq^{\mathbf{I}} \mathcal{L}_2$ and $\mathcal{L}_2 \leq^{\mathbf{I}} \mathcal{L}_1$.

The reduction defined above was called strong-reduction in [JS96]. The above reduction without clause (c) was called weak-reduction. Since in this paper we will be only concerned with strong-reductions, we just refer to them as reductions. (Weak reductions are not "sharp" enough to provide real distinction between most of the classes considered in this paper).

Intuitively, $\mathcal{L}_1 \leq^{\mathbf{I}} \mathcal{L}_2$ just in case there exists an operator Θ that transforms texts for languages in $\mathcal{L}_1$ into texts for languages in $\mathcal{L}_2$ and there exists another operator Ψ that behaves as follows: if Θ transforms text T (for a language in $\mathcal{L}_1$) to text T' (for a language in $\mathcal{L}_2$), then Ψ transforms $\mathbf{I}$-admissible sequences for T' into $\mathbf{I}$-admissible sequences for T. Thus, informally, the operator Ψ has "to work" only on $\mathbf{I}$-admissible sequences for such texts T'. In other words, if α is a sequence of grammars which is *not* $\mathbf{I}$-admissible for any text T' in $\{\Theta(T) \mid \text{content}(T) \in \mathcal{L}_1\}$, then $\Psi(\alpha)$ can be defined *arbitrarily*. This property will be used implicitly at all places below where we have to define operators Ψ witnessing (together with operators Θ) some reducibility. Note that this approach both simplifies the corresponding definitions and preserves the computability of the so defined operators.

Additionally different texts for some language $L \in \mathcal{L}_1$, are transformed into (possibly different) texts for same language $L' \in \mathcal{L}_2$.

Intuitively, for many identification criterion $\mathbf{I}$ such as $\mathbf{TxtEx}$, if $\mathcal{L}_1 \leq^{\mathbf{I}} \mathcal{L}_2$ then the problem of identifying $\mathcal{L}_2$ in the sense of $\mathbf{I}$ is at least as hard as the problem of identifying $\mathcal{L}_1$ in the sense of $\mathbf{I}$, since the solvability of the former problem implies the solvability of the latter one. That is given any machine $\mathbf{M}_2$ which $\mathbf{I}$-identifies $\mathcal{L}_2$, one can construct a machine $\mathbf{M}_1$ which $\mathbf{I}$-identifies $\mathcal{L}_1$. To see this, for $\mathbf{I} = \mathbf{TxtEx}$, suppose Θ and Ψ witness $\mathcal{L}_1 \leq^{\mathbf{I}} \mathcal{L}_2$. $\mathbf{M}_1(T)$, for a text T

is defined as follows. Let $p_n = \mathbf{M}_2(\Theta(T)[n])$, and $\alpha = p_0, p_1, \ldots$. Let $\alpha' = \Psi(\alpha) = p'_0, p'_1, \ldots$. Then let $\mathbf{M}_1(T) = \lim_{n\to\infty} p'_n$. Consequently, $\mathcal{L}_2$ may be considered as a "hardest" problem for $\mathbf{I}$-identification if for *all* classes $\mathcal{L}_1 \in \mathbf{I}$, $\mathcal{L}_1 \leq^{\mathbf{I}} \mathcal{L}_2$ holds. If $\mathcal{L}_2$ itself belongs to $\mathbf{I}$, then $\mathcal{L}_2$ is said to be *complete*. We now formally define these notions of hardness and completeness for the above reduction.

Definition 6 [JS96] Let $\mathbf{I}$ be an identification criterion. Let $\mathcal{L} \subseteq \mathcal{E}$ be given.
(a) If for all $\mathcal{L}' \in \mathbf{I}$, $\mathcal{L}' \leq^{\mathbf{I}} \mathcal{L}$, then $\mathcal{L}$ is $\leq^{\mathbf{I}}$*-hard*.
(b) If $\mathcal{L}$ is $\leq^{\mathbf{I}}$-hard and $\mathcal{L} \in \mathbf{I}$, then $\mathcal{L}$ is $\leq^{\mathbf{I}}$*-complete*.

Proposition 1 *([JS96])* $\leq^{\mathbf{TxtEx}}$ *is reflexive and transitive.*

The above proposition holds for most natural learning criteria.

Proposition 2 *(based on [JS97]) Suppose* $\mathcal{L} \leq^{\mathbf{I}} \mathcal{L}'$*, via* Θ *and* Ψ*. Then, for all* $L, L' \in \mathcal{L}$*,* $L \subseteq L' \Rightarrow \Theta(L) \subseteq \Theta(L')$*.*

We will be using Proposition 2 implicitly when we are dealing with reductions. Since, for $\mathcal{L} \leq^{\mathbf{I}} \mathcal{L}'$ via Θ and Ψ, for all $L \in \mathcal{L}$, $\Theta(L)$ is defined (= some $L' \in \mathcal{L}'$), when considering reductions, we often consider Θ as mapping finite sets to (possibly infinite) sets instead of mapping sequences to sequences.[1] This is clearly without loss of generality, as one can easily convert such Θ to Θ as in Definition 5 of reduction.

4 *HALF* and *Q*-classes

In this section we introduce classes of languages and corresponding degrees of intrinsic complexity that form the "scale" being used for estimating complexity of learning open semi-hulls and open hulls. First we define "ground" natural classes that are being used as "bricks" to build our hierarchy of degrees.

Definition 7 $\mathit{INIT} = \{L \subseteq N \mid (\exists i \in N)[L = \{x \in N \mid x \leq i\}]\}$.
$\mathit{COINIT} = \{L \subseteq N \mid (\exists i \in N)[L = \{x \in N \mid x \geq i\}]\}$.
$\mathit{HALF} = \{L \subseteq Z \mid (\exists i \in Z)[L = \{x \in Z \mid x \geq i\}]\}$.

Note that officially our definition for languages and r.e. sets as in the Section 2, only allows subsets of N. Since, one can easily code Z onto N, by slight abuse of convention, we can consider subsets of Z also as languages. We thus assume an implicit coding of Z onto N whenever we deal with languages and language classes involving *HALF*, without explicitly stating so.

While both classes *INIT* and *COINIT* are monotonically learnable, the types of conjectures being used to learn *INIT* and, respectively, *COINIT* are obviously different. For *INIT* the maximum element in the input gives a "code" for the language, whereas in *COINIT* the minimum element gives the code. Note that the maximum element used in *INIT* strategy is unbounded, whereas the minimum element for *COINIT* is bounded by 0. So, not surprisingly, the degrees of *INIT* and *COINIT* were proven in [JS96] to be different.

Classes *HALF* and *COINIT* are learnable by similar strategies, however, the minimum element in *HALF* is "unbounded". We will formally prove below that *HALF* is different from both *INIT* and *COINIT*. Furthermore, we will show that *HALF* can be viewed as a cross product of *INIT* and *COINIT*.

[1]For infinite X, by definition, $\Theta(X)$ would be $\bigcup_{X' \subseteq X : \mathrm{card}(X') < \infty} \Theta(X')$.

There are several other natural classes considered in the literature such as *FINITE* (which is equivalent to *INIT*), *SINGLE*, *COSINGLE*, etc. but we will not be concerned with them here since they will not be relevant to our results.

Now we define the cross product of arbitrary classes $\mathcal{L}_1$ and $\mathcal{L}_2$.

Definition 8 Let $\mathcal{L}_1, \mathcal{L}_2$ be two classes of languages. Then $\mathcal{L}_1 \times \mathcal{L}_2 = \{L_1 \times L_2 \mid L_1 \in \mathcal{L}_1, L_2 \in \mathcal{L}_2\}$.

This definition can be naturally extended to any finite number of dimensions. For example, one can naturally define $\mathcal{L}_1 \times \mathcal{L}_2 \times \mathcal{L}_3$, etc.

The following propositions trivially follow from the above definitions.

Proposition 3 $\mathcal{L}_1 \leq^{\mathbf{TxtEx}} \mathcal{L}_1 \times \mathcal{L}_2$.

Proposition 4 *Suppose* $\mathcal{L}_1 \leq^{\mathbf{TxtEx}} \mathcal{L}_2$. *Then* $\mathcal{L}_1 \times \mathcal{L}_3 \leq^{\mathbf{TxtEx}} \mathcal{L}_2 \times \mathcal{L}_3$.

Proposition 5 $\mathcal{L}_1 \times \mathcal{L}_2 \equiv^{\mathbf{TxtEx}} \mathcal{L}_2 \times \mathcal{L}_1$.

Proposition 6 $(\mathcal{L}_1 \times \mathcal{L}_2) \times \mathcal{L}_3 \equiv^{\mathbf{TxtEx}} \mathcal{L}_1 \times (\mathcal{L}_2 \times \mathcal{L}_3) \equiv^{\mathbf{TxtEx}} \mathcal{L}_1 \times \mathcal{L}_2 \times \mathcal{L}_3$.

Thus, degrees formed using cross product are commutative and associative.

Proposition 7 *[JKW99] INIT* $\times$ *INIT* $\equiv^{\mathbf{TxtEx}}$ *INIT*.

Proposition 8 *[JKW99] COINIT* $\times$ *COINIT* $\equiv^{\mathbf{TxtEx}}$ *COINIT*

Proposition 9 *[JKW99] INIT* $\times$ *COINIT* $\not\leq^{\mathbf{TxtEx}}$ *INIT and INIT* $\times$ *COINIT* $\not\leq^{\mathbf{TxtEx}}$ *COINIT*.

Theorem 1 *HALF* $\equiv^{\mathbf{TxtEx}}$ *INIT* $\times$ *COINIT*.

Proof. For $a \in Z$, let $L_a = \{x \in Z \mid x \geq a\}$. For $i, j \in N$, let $X_{i,j} = \{x \in N \mid x \leq i\} \times \{x \mid x \geq j\}$.

We now define Θ and Ψ witnessing that *HALF* $\leq^{\mathbf{TxtEx}}$ *INIT* $\times$ *COINIT*.

For any finite subset Y of Z, let $\Theta(Y) = \bigcup_{a \geq 0, a \in Y} X_{0,a} \cup \bigcup_{a<0, a \in Y} X_{-a,0}$.

It can be easily verified that $\Theta(L_a) = X_{0,a}$ (if $a \geq 0$), and $\Theta(L_a) = X_{-a,0}$ (if $a < 0$).

Ψ is defined as follows. Suppose a sequence α of grammars converges to grammar p. Suppose $i = \max(\{x \in N \mid (\exists y \in N)[\langle x, y\rangle \in W_p]\})$, and $j = \min(\{y \in N \mid (\exists x \in N)[\langle x, y\rangle \in W_p]\})$. Then, if $i = 0$, then $\Psi(\alpha)$ converges to a program for L_j. If $i > 0$, then $\Psi(\alpha)$ converges to a program for L_{-i}.

It is easy to verify that Θ and Ψ witness that *HALF* $\leq^{\mathbf{TxtEx}}$ *INIT* $\times$ *COINIT*.

Now, we show that *INIT* $\times$ *COINIT* $\leq^{\mathbf{TxtEx}}$ *HALF*.

Define h as follows:

for $i, j \in N$,

$h(i, j) = -[(i+1)(i+2)/2] + 1 + j$, if $i \geq j$;

$h(i, j) = [j(j+1)/2] - i$, if $i < j$.

Intuitively picture $h(i, j)$ as follows:

$\cdots$	$h(2,0)$	$h(2,1)$	$h(2,2)$	$h(1,0)$	$h(1,1)$	$h(0,0)$
$\cdots$	-5	-4	-3	-2	-1	0

$h(0,1)$	$h(1,2)$	$h(0,2)$	$h(2,3)$	$h(1,3)$	$h(0,3)$	$\cdots$
1	2	3	4	5	6	$\cdots$

Note that for all $i, j, k, l \in N$, if $i \geq k$ and $j \leq l$, then $h(i,j) \leq h(k,l)$. Now, for any set $Y \subseteq N^2$, let $\Theta(Y) = \bigcup_{\langle i,j\rangle \in Y} L_{h(i,j)}$. It is easy to verify that $\Theta(X_{i,j}) = L_{h(i,j)}$.

$\Psi(\alpha)$ is defined as follows. If a sequence α of grammars converges to a grammar p for $L_{h(i,j)}$, then $\Psi(\alpha)$ converges to a grammar for $X_{i,j}$. (Note that, if p, is a grammar for some $L_{h(i,j)}$, then such i, j can be determined in the limit from p.) It is easy to verify that Θ and Ψ witness that $\mathit{INIT} \times \mathit{COINIT} \leq^{\mathbf{TxtEx}} \mathit{HALF}$. ∎

Now, following [JKW99], we are going to combine classes *INIT*, *COINIT*, and *HALF* to form classes of "multidimensional" languages, where, to learn the "dimension" L_{k+1} of a language L, the learner must first learn the "parameters" $i_1, \ldots, i_k$ of the "dimensions" $L_1, \ldots, L_k$; then L_{k+1} is the projection $\{x_{k+1} | \langle i_1, \ldots, i_k, x_{k+1}, x_{k+2}, \ldots, x_n\rangle \in L\}$ with a simple sublanguage whose description is specified yet by $i_1, \ldots, i_k$. Once it has been determined which projection must be learned, the learner can use a predefined *INIT*, *COINIT*, or *HALF*-type strategy to learn the projection in question. For example, one can consider a class of "two-dimensional" languages $(\mathit{INIT}, \mathit{COINIT})$, where the first "dimension" $L_1 = \{x | \langle x, y\rangle \in L\}$ of any language L belongs to *INIT*, and if i is the "parameter" describing L_1 (that can be learned by an *INIT*-type strategy) then the projection $\{y | \langle i, y\rangle \in L\}$ is in *COINIT*.

Below for any tuples X and Y, let $X \cdot Y$ denote the concatenation of X and Y. That is if $X = \langle x_1, x_2, \ldots, x_n\rangle$ and $Y = \langle y_1, y_2, \ldots, y_m\rangle$ then $X \cdot Y = \langle x_1 \ldots, x_n, y_1, \ldots y_m\rangle$. Let $\mathit{BASIC} = \{\mathit{INIT}, \mathit{COINIT}, \mathit{HALF}\}$.

In part (c) of the following defintion and in later situations in languages involving *HALF* we sometimes abuse notation slightly and allow elements of Z as components of the pairing function $\langle \cdots \rangle$. This is for ease of notation, and one could easily replace these by using some coding of Z onto N.

Definition 9 Suppose $k \geq 1$. Let $Q \in \mathit{BASIC}^k$. Let $I \in N^k$. Then inductively on k, we define the languages L_I^Q and $T(L_I^Q)$ and $P(L_I^Q)$ as follows.
If $k = 1$, then
(a) if $Q = (\mathit{INIT})$ and $I = (i)$, $i \in N$, then
$T(L_I^Q) = \{\langle x\rangle \mid x < i\}$, $P(L_I^Q) = \{\langle i\rangle\}$, and $L_I^Q = T(L_I^Q) \bigcup P(L_I^Q)$.
(b) if $Q = (\mathit{COINIT})$ and $I = (i)$, $i \in N$, then
$T(L_I^Q) = \{\langle x\rangle \mid x > i\}$, $P(L_I^Q) = \{\langle i\rangle\}$, and $L_I^Q = T(L_i^Q) \bigcup P(L_i^Q)$.
(c) if $Q = (\mathit{HALF})$ and $I = (i)$, $i \in Z$, then
$T(L_I^Q) = \{\langle x\rangle \mid x > i\}$, $P(L_I^Q) = \{\langle i\rangle\}$, and $L_I^Q = T(L_i^Q) \bigcup P(L_i^Q)$.
Now suppose we have already defined L_I^Q for $k \leq n$. We then define L_I^Q for $k = n + 1$ as follows. Suppose $Q = (q_1, \ldots, q_{n+1})$ and $I = (i_1, \ldots, i_{n+1})$. Let $Q_1 = (q_1)$ and $Q_2 = (q_2, \ldots, q_{n+1})$. Let $I_1 = (i_1)$ and $I_2 = (i_2, \ldots, i_{n+1})$. Then,
$T(L_I^Q) = \{X \cdot Y \mid X \in T(L_{I_1}^{Q_1})$, OR $X \in P(L_{I_1}^{Q_1})$ and $Y \in T(L_{I_2}^{Q_2})\}$,
$P(L_I^Q) = \{X \cdot Y \mid X \in P(L_{I_1}^{Q_1})$ and $Y \in P(L_{I_2}^{Q_2})\}$, and
$L_I^Q = T(L_I^Q) \bigcup P(L_I^Q)$.

Intuitively, in the above definition $T(L_I^Q)$ denotes the "terminating" part of the language that is specified yet by $i_1, \ldots, i_n$ (and, thus, can be learned trivially), and $P(L_I^Q)$ denotes the "propagating" part of the language L_I^Q that could be used for adding a language in dimension "$n + 1$." (See [JKW99] for more details and motivation on the terminology of terminating and propagating.)

For ease of notation we often write $L^Q_{(i_1, i_2, \ldots, i_k)}$ as $L^Q_{i_1, i_2, \ldots, i_k}$.

Definition 10 *Let $k \geq 1$. Let $Q = (q_1, \ldots, q_k) \in BASIC^k$ and $R = R_1 \times R_2 \times \cdots \times R_k$, where for $1 \leq i \leq k$, $R_i \subseteq N$ if $q_i \in \{INIT, COINIT\}$, and $R_i \subseteq Z$, if $q_i = HALF$. Then the class $\mathcal{L}^{Q,R}$ is defined as*
$\mathcal{L}^{Q,R} = \{L_I^Q \mid I \in R\}$.

For technical convenience, for $Q = ()$, $I = ()$, $R = \{I\}$, we also define $T(L_I^Q) = \emptyset$, $P(L_Q^I) = \{\langle\rangle\}$, and $L_I^Q = T(L_I^Q) \bigcup P(L_I^Q)$, and $\mathcal{L}^{Q,R} = \{L_I^Q\}$.

Note that we have used a slightly different notation for defining the classes $\mathcal{L}^{Q,R}$ (for example instead of $INIT$, we now use $\mathcal{L}^{(INIT),N)}$). This is for clarity of notation.

Also, our main interest is for R_i's being N or Z (based on whether $q_i \in \{INIT, COINIT\}$ or $q_i = HALF$), though (as the following proposition shows) it doesn't matter as long as R_i is (or contains) an infinite recursive subset of N, if $q_i \in \{INIT, COINIT\}$, and R_i is (or contains) an infinite recursive subset of Z with infinite intersection with both N and Z^-, if $q_i = HALF$. The usage of general R is more for ease of proving some of our theorems.

Proposition 10 *Suppose $k \geq 1$. Let $Q \in BASIC^k$. Let $R = R_1 \times R_2 \times \cdots \times R_k$, where each R_i is an infinite recursive subset of N, if $q_i \in \{INIT, COINIT\}$ and R_i is a recursive subset of Z, with infinite intersection with both N and Z^-, if $q_i = HALF$. Let $R' = R'_1 \times R'_2 \times \cdots \times R'_k$, where R'_i is N, if $q_i \in \{INIT, COINIT\}$ and R'_i is Z, if $q_i = HALF$. Then, $\mathcal{L}^{Q,R} \equiv^{\mathbf{TxtEx}} \mathcal{L}^{Q,R'}$.*

For ease of notation, if $Q = (q_1, q_2, \ldots, q_n)$ and $R = R_1 \times R_2 \times \ldots \times R_n$, where $R_i = N$ if $q_i \in \{INIT, COINIT\}$ and $R_i = Z$ if $q_i = HALF$, then we drop R from $\mathcal{L}^{Q,R}$, using just $\mathcal{L}^Q$.

The immediate question is which classes Q represent different strong degrees.

Proposition 11 *Suppose $Q = (q_1, \ldots, q_{k-1}, q_k, q_{k+1}, \ldots, q_l)$, and $Q' = (q_1, \ldots, q_{k-1}, q', q_{k+1}, \ldots, q_l)$, where $q_k \in \{INIT, COINIT\}$ and $q' = HALF$, and other $q_i \in BASIC$.*
Then, $\mathcal{L}^Q \leq^{\mathbf{TxtEx}} \mathcal{L}^{Q'}$.

Since $INIT \times COINIT \leq^{\mathbf{TxtEx}} \mathcal{L}^Q$, for $Q = (INIT, COINIT)$, or $Q = (COINIT, INIT)$ [JKW99], we have

Proposition 12 *Suppose*
$Q = (q_1, \ldots, q_{k-1}, HALF, q_{k+1}, \ldots, q_n)$, and $Q' = (q_1, \ldots, q_{k-1}, INIT, COINIT, q_{k+1}, \ldots, q_n)$ or $Q' = (q_1, \ldots, q_{k-1}, COINIT, INIT, q_{k+1}, \ldots, q_n)$. Then $\mathcal{L}^Q \leq^{\mathbf{TxtEx}} \mathcal{L}^{Q'}$.

Proposition 13 *(Based on [JS96]) Suppose $Q = (INIT)$, $R = R_1$, $Q' = (COINIT)$, and $R' = R'_1$, where R_1 and R'_1 are infinite subsets of N. Then $\mathcal{L}^{Q,R} \not\leq^{\mathbf{TxtEx}} \mathcal{L}^{Q',R'}$, and*
$\mathcal{L}^{Q',R'} \not\leq^{\mathbf{TxtEx}} \mathcal{L}^{Q,R}$.

The following technical definition introduces an ordering on all k-tuples of parameters $i_1, \ldots, i_k$ of languages in an arbitrary class $\mathcal{L}^Q$. This ordering will be helpful for the result in the next section.

Definition 11 Suppose $Q = (q_1, \ldots, q_k)$, where each $q_i \in \{HALF, INIT, COINIT\}$, for $1 \leq i \leq k$. Let $Q' = (q_2, \ldots, q_k)$. We say that $\langle i_1, \ldots, i_k\rangle \leq_Q \langle j_1, \ldots, j_k\rangle$ iff

(a) if $q_1 = INIT$, then $[i_1 < j_1]$ or $[i_1 = j_1$ and $\langle i_2, \ldots, i_k\rangle \leq_{Q'} \langle j_2, \ldots, j_k\rangle]$;

(b) if $q_1 = COINIT$ or $q_1 = HALF$, then $[i_1 > j_1]$ or $[i_1 = j_1$ and $\langle i_2, \ldots, i_k\rangle \leq_{Q'} \langle j_2, \ldots, j_k\rangle]$.
Here, for $Q = ()$, we assume that $\langle\rangle \leq_Q \langle\rangle$.

5 Q-hierarchy Involving *HALF*

In this section we establish a hierarchy among Q-classes that will serve as a "scale" for complexity of our geometrical concepts.

Definition 12 Q is said to be a pseudo-subsequence of Q', iff there exists a subsequence Q'' of Q', such that Q'' is obtainable from Q by
(i) replacing some *INIT* with *HALF*,
(ii) replacing some *COINIT* with *HALF*,
(iii) replacing some *HALF* with $(\mathit{COINIT}, \mathit{INIT})$, or
(iv) replacing some *HALF* with $(\mathit{INIT}, \mathit{COINIT})$.

Proposition 14 *Suppose $Q, Q', Q'' \in \mathit{BASIC}^*$. Then if Q is a pseudo-subsequence of Q' and Q' is a pseudo-subsequence of Q'', then Q is a pseudo-subsequence of Q''.*

Proof. Follows from definition of pseudo-subsequence. ∎

Proposition 15 *Suppose $Q, Q' \in \mathit{BASIC}^*$ and Q is a pseudo-subsequence of Q'. Then $\mathcal{L}^Q \leq^{\mathbf{TxtEx}} \mathcal{L}^{Q'}$.*

Proof. Follows from definition of pseudo-subsequence, Proposition 11, and Proposition 12. ∎

Proposition 16 *Suppose $Q = (q_1, q_2, \ldots, q_k)$ and $Q' = (q'_1, q'_2, \ldots, q'_l)$, where each $q_i, q'_i \in$ BASIC.*

(a) If $q_1 = q'_1$, then $(q_2, \ldots, q_k)$ is a pseudo-subsequence of $(q'_2, \ldots, q'_l)$, iff Q is a pseudo-subsequence of Q'.

(b) Suppose $q_1, q'_1 \in \{\mathit{INIT}, \mathit{COINIT}\}$, but $q_1 \neq q'_1$. Then, Q is a pseudo-subsequence of Q' iff Q is a pseudo-subsequence of $Q'' = (q'_2, q'_3, \ldots, q'_l)$.

(c) Suppose $q_1 = \mathit{HALF}$. Then, Q is a pseudo-subsequence of Q' implies $(\mathit{INIT}, q_2, \ldots, q_k)$ is a pseudo-subsequence of Q'.

(d) Suppose $q_1 = \mathit{HALF}$. Then, Q is a pseudo-subsequence of Q' implies $(\mathit{COINIT}, q_2, \ldots, q_k)$ is a pseudo-subsequence of Q'.

(e) Suppose $q'_1 = \mathit{HALF}$. Then, Q is a pseudo-subsequence of Q' iff $(q_2, q_3, \ldots, q_k)$ is a pseudo-subsequence of $(q'_2, q'_3, \ldots, q'_l)$.

Proof. Follows from the definition of pseudo-subsequence. ∎

Proposition 17 *Suppose $Q = (q_1, q_2, \ldots, q_k)$ and $Q' = (q'_1, q'_2, \ldots, q'_l)$, where each $q_i, q'_i \in$ BASIC. Suppose Q is not a pseudo-subsequence of Q'.*

(a) If $q_1 = q'_1 \in \{\mathit{INIT}, \mathit{COINIT}\}$, $q_2 \in \{\mathit{INIT}, \mathit{COINIT}\}$, and $q_2 \neq q_1$, then $Q'' = (q_2, q_3, \ldots, q_k)$ is not a pseudo-subsequence of Q'.

(b) If $q_1 = \mathit{HALF}$, and $q'_1 = \mathit{INIT}$, then $Q'' = (\mathit{COINIT}, q_2, \ldots, q_k)$ is not a pseudo-subsequence of Q'.

(c) If $q_1 = \mathit{HALF}$, and $q'_1 = \mathit{COINIT}$, then $Q'' = (\mathit{INIT}, q_2, \ldots, q_k)$ is not a pseudo-subsequence of Q'.

(d) If $q_1 = q'_1 \in \{\mathit{INIT}, \mathit{COINIT}\}$, $q_2 = \mathit{HALF}$, $l \geq 2$, and $q'_2 \in \{\mathit{INIT}, \mathit{HALF}\}$, then $Q'' = (q_1, \mathit{COINIT}, q_3, q_4, \ldots, q_k)$ is not a pseudo-subsequence of Q'.

(e) If $q_1 = q'_1 \in \{\mathit{INIT}, \mathit{COINIT}\}$, $q_2 = \mathit{HALF}$, $l \geq 2$, and $q'_2 \in \{\mathit{COINIT}, \mathit{HALF}\}$, then $Q'' = (q_1, \mathit{INIT}, q_3, q_4, \ldots, q_k)$ is not a pseudo-subsequence of Q'.

PROOF. We show part (a), (b) and (e). Proof of part (c) is similar to part (b) and proof of part (e) is similar to part (d).

(a) Let $Q''' = (q'_2, q'_3, \ldots, q'_l)$. Now by Proposition 16(a) Q is a pseudo-subsequence of Q' iff Q'' is pseudo-subsequence of Q'''. By Proposition 16(b) Q'' is pseudo-subsequence of Q''' iff Q'' is pseudo-subsequence of Q'. Thus, since Q is not a pseudo-subsequence of Q' it follows that Q'' is not a pseudo-subsequence a pseudo-subsequence of Q'.

(b) Suppose by way of contradiction that $(COINIT, q_2, \ldots, q_k)$ is a pseudo-subsequence of Q'. Then by applying Proposition 16(b), we have that $(COINIT, q_2, \ldots, q_k)$ is a pseudo-subsequence of $(q'_2, q'_3, \ldots, q'_l)$. Thus by Proposition 16(a) we have $(INIT, COINIT, q_2, \ldots, q_k)$ is a pseudo-subsequence of $(q'_1, q'_2, \ldots, q'_l)$. But since $(HALF, q_2, \ldots, q_k)$ is a pseudo-subsequence of $(INIT, COINIT, q_2, \ldots, q_k)$ we have, $Q = (HALF, q_2, \ldots, q_k)$ is a pseudo-subsequence of $Q' = (q'_1, q'_2, q'_3, q'_4, \ldots, q'_l)$. A contradiction to the hypothesis. Part (b) follows.

(d) Let $Q''' = (q'_2, q'_3, \ldots, q'_l)$. We consider two cases.

Case 1: $q'_2 = HALF$.

By Proposition 16(a), Q is a pseudo-subsequence of Q' iff $(q_2, \ldots, q_k)$ is a pseudo-subsequence of Q'''. By Proposition 16(e), $(q_2, q_3, \ldots, q_k)$ is a pseudo-subsequence of Q''', iff $(COINIT, q_3, \ldots, q_k)$ is a pseudo-subsequence of Q'''. Thus, since Q is not a pseudo-subsequence of Q' it follows that $(COINIT, q_3, \ldots, q_k)$ is not a pseudo-subsequence a pseudo-subsequence of Q'''. Thus, by Proposition 16(a), it follows that $(q_1, COINIT, q_3, \ldots, q_k)$ is not a pseudo-subsequence a pseudo-subsequence of Q'.

Case 2: $q'_2 = INIT$.

By hypothesis and Proposition 16(a) we have that $(q_2, q_3, \ldots, q_k)$ is not a pseudo-subsequence of $(q'_2, q'_3, \ldots, q'_l)$. Thus by part (b) $(COINIT, q_3, \ldots, q_k)$ is not a pseudo-subsequence of $(q'_2, q'_3, \ldots, q'_l)$. Thus by Proposition 16(a) $(q_1, COINIT, q_3, \ldots, q_k)$ is not a pseudo-subsequence of Q'. ∎

Theorem 2 *Suppose $Q = (q_1, \ldots, q_k) \in BASIC^k$ and $Q' = (q'_1, \ldots, q'_l) \in BASIC^l$. Let $R = R_1 \times R_2 \times \cdots \times R_k$, $R' = R'_1 \times R'_2 \times \cdots \times R'_l$, where each R_i (R'_i) is an infinite subset of N, if $q_i \in \{INIT, COINIT\}$ ($q'_i \in \{INIT, COINIT\}$), and R_i (R'_i) is a subset of Z, with infinite intersection with both N and Z^-, if $q_i = HALF$ ($q'_i = HALF$).*

If Q is not a pseudo-subsequence of Q' then $\mathcal{L}^{Q,R} \not\leq^{\mathbf{TxtEx}} \mathcal{L}^{Q',R'}$.

PROOF. We prove the theorem by double induction (first on k and then on l). For $k = 0$ or $l = 0$ theorem clearly holds. Suppose by induction that the theorem holds for $k \leq m$, $l \in N$, and for $k = m+1$, $l \leq r$. We then show that the theorem holds for $k = m+1$ and $l = r+1$. Suppose by way of contradiction that Θ (along with Ψ) witnesses that $\mathcal{L}^{Q,R} \leq^{\mathbf{TxtEx}} \mathcal{L}^{Q',R'}$.

We consider the following cases:

Case 1: $q_1 = INIT$.

Case 1.1: $q'_1 = COINIT$.

Consider σ, which minimizes $i \in N$ such that $\langle i, \ldots \rangle \in \text{content}(\Theta(\sigma))$. Let $j \in N$ be the maximum number such that $\langle j, \ldots \rangle \in \text{content}(\sigma)$. It follows that, for any $j' > j$, $j' \in R_1$, $\Theta(L^Q_{j',\ldots})$, for any values of other parameters, is of the form $L^{Q'}_{i,\ldots}$, for some values of the other parameters. Thus, Θ (along with Ψ) essentially witnesses that $\mathcal{L}^{Q,RR} \leq^{\mathbf{TxtEx}} \mathcal{L}^{QQ',RR'}$, where RR is obtained from R by replacing R_1 by $R_1 - \{x \mid x \leq j\}$, and QQ' is obtained from Q' by dropping q'_1 and RR' is obtained from R' by dropping R'_1. Now we are done by induction hypothesis.

Case 1.2: $q'_1 = INIT$.

In this case $k \geq 2$.

Case 1.2.1: $q_2 = INIT$.

Fix $i_1 \in R_1$, and consider the set $\bigcup_{i_2 \in R_2,\ldots} \Theta(L^Q_{i_1,i_2,\ldots})$. Suppose this set contains $\langle i'_1, \ldots\rangle$, for arbitrarily large i'_1. Then for any $ii_1 > i_1$ (since $L^Q_{ii_1,\ldots} \supseteq L^Q_{i_1,\ldots}$ for all possible values of other parameters) we have that $\Theta(L^Q_{ii_1,\min(R_2),\min(R_3),\ldots})$ contains elements of form $\langle i'_1, \ldots\rangle$ for arbitrarily large i'_1. Thus, $\Theta(L^Q_{ii_1,\min(R_2),\min(R_3),\ldots}) \notin \mathcal{L}^{Q'}$.

So let i'_1 be maximum value such that some element of form $\langle i'_1, \ldots\rangle$ is in $\bigcup_{i_2 \in R_2,\ldots} \Theta(L^Q_{i_1,i_2,\ldots})$.

Let σ be such that content$(\sigma) \subseteq L^Q_{i_1,i_2,\ldots}$, and $\Theta(\sigma)$ contains an element of form $\langle i'_1, \ldots\rangle$. Let i_2 be maximum value such that some element of form $\langle i_1, i_2, \ldots\rangle$ is in content(σ). It follows that, for all $ii_2 > i_2$, $ii_2 \in R_2$, $\Theta(L^Q_{i_1,ii_2,\ldots})$, for any value of other parameters, is of form $L^{Q'}_{i'_1,\ldots}$, for some value of other parameters. Thus, Θ (along with Ψ) essentially witnesses that $\mathcal{L}^{QQ,RR} \leq^{\mathbf{TxtEx}} \mathcal{L}^{QQ',RR'}$, where QQ is obtained from Q by dropping q_1, QQ' is obtained from Q' by dropping q'_1, RR' is obtained from R' by dropping R'_1 and RR is obtained from R by dropping R_1 plus changing R_2 to $R_2 - \{x \mid x \leq i_2\}$. Now we are done by induction hypothesis.

Case 1.2.2: $q_2 = COINIT$.

In this case, by Proposition 17(a) QQ obtained from Q by dropping q_1 is not a pseudo-subsequence of Q'. Thus we are done by induction hypothesis.

Case 1.2.3: $q_2 = HALF$.

If $l = 1$, then we are done. So assume $l \geq 2$.

Case 1.2.3.1: $q'_2 = INIT$ or $HALF$.

Then, by replacing q_2 by $COINIT$, using Proposition 17(d) we still have that Q is not a pseudo-subsequence of Q'. Thus, we can use Case 1.2.2.

Case 1.2.3.2: $q'_2 = COINIT$.

Then, by replacing q_2 by $INIT$, using Proposition 17(e) we still have that Q is not a pseudo-subsequence of Q'. Thus, we can use Case 1.2.1.

Case 1.3 $q'_1 = HALF$.

In this case, let $i \in R_1$. Suppose $\Theta(L^Q_{i,\ldots}) = L^{Q'}_{j,\ldots}$, for some values of other parameters. But then for all $i' > i$, $\Theta(L^Q_{i',\ldots})$, for any value of other parameters, must be of form $L^{Q'}_{j',\ldots}$, for some value of other parameters, where $j' \leq j$. Thus, one could essentially consider Θ (along with Ψ) as a reduction from $\mathcal{L}^{Q,RR}$ to $\mathcal{L}^{QQ',RR'}$, where RR is obtained from R by replacing R_1 by $R_1 - \{x \mid x \leq i\}$, QQ' is obtained from Q' by replacing q'_1 with $INIT$, and RR' is obtained from R' by replacing R'_1 by $\{x \in N \mid -x + j \in R'_1 - \{y \mid y \geq j\}\}$. Thus we can use Case 1.2.

Case 2: $q_1 = COINIT$. This case is very similar to Case 1. We give the analysis for completeness sake.

Case 2.1: $q'_1 = INIT$.

Let $i \in N$ be minimum value such that $\Theta(L^Q_{\ldots}) = L^{Q'}_{i,\ldots}$, for some values of the other parameters. Let $j \in R_1$ be such that $\Theta(L^Q_{j,\ldots}) = L^{Q'}_{i,\ldots}$, for some values of the parameters. It follows that for all $j' > j$, $j' \in R_1$, $\Theta(L^Q_{j',\ldots})$, for any value of other parameters, is of form $L^{Q'}_{i,\ldots}$, for some value of other parameters.

Thus, Θ (along with Ψ) essentially witnesses that $\mathcal{L}^{Q,RR} \leq^{\mathbf{TxtEx}} \mathcal{L}^{QQ',RR'}$, where RR is obtained from R by replacing R_1 by $R_1 - \{x \mid x \leq j\}$, and QQ' is obtained from Q' by dropping q'_1 and RR' is obtained from R' by dropping R'_1. Now we are done by induction hypothesis.

Case 2.2: $q'_1 = COINIT$.

In this case $k \geq 2$.

Case 2.2.1: $q_2 = INIT$.

In this case, by Proposition 17(a) QQ obtained from Q by dropping q_1 is not a pseudo-subsequence of Q'. Thus we are done by induction hypothesis.

Case 2.2.2: $q_2 = COINIT$. Fix $i_1 \in R_1$, and consider $\Theta(L^Q_{i_1,\ldots}) = L^{Q'}_{i'_1,\ldots}$. If i'_1 achieves arbitrary high value (for some values of other parameters) then, for $ii_1 > i_1$, since $L^Q_{ii_1,0,0,\ldots} \subseteq L^Q_{i_1,\ldots}$, and $\bigcap_{i'_1 \in R'_1} L^{Q'}_{i'_1,\ldots} = \emptyset$, $\Theta(L_{ii_1,0,0,\ldots}) = \emptyset \notin \mathcal{L}^{Q',R'}$.

So let i'_1 be maximum value such that for some value of other parameters, $\Theta(L^{Q,R}_{i_1,\ldots}) = L^{Q',R'}_{i'_1,\ldots}$. let i_2 be such that, for some value of other parameters, $\Theta(L^{Q,R}_{i_1,i_2,\ldots}) = L^{Q',R'}_{i'_1,\ldots}$. It follows that, for all $ii_2 > i_2$, $\Theta(L^{Q,R}_{i_1,ii_2,\ldots})$ is of form $L^{Q',R'}_{i'_1,\ldots}$. Thus, Θ (along with Ψ) essentially witnesses that $\mathcal{L}^{QQ,RR} \leq^{\mathbf{TxtEx}} \mathcal{L}^{QQ',RR'}$, where QQ is obtained from Q by dropping q_1, QQ' is obtained from Q' by dropping q'_1, RR' is obtained from R' by dropping R'_1 and RR is obtained from R by dropping R_1 plus changing R_2 to $R_2 - \{x \mid x \leq i_2\}$. Now we are done by induction hypothesis.

Case 2.2.3: $q_2 = HALF$.

If $l = 1$, then we are done. So assume $l \geq 2$.

Case 2.2.3.1: $q'_2 = INIT$ or $HALF$.

Then, by replacing q_2 by $COINIT$, using Proposition 17(d), we still have that Q is not a pseudo-subsequence of Q'. Thus, we can use Case 2.2.2.

Case 2.2.3.2: $q'_2 = COINIT$.

Then, by replacing q_2 by $INIT$, using Proposition 17(e), we still have that Q is not a pseudo-subsequence of Q'. Thus, we can use 2.2.1.

Case 3: $q_1 = HALF$.

Case 3.1: $q'_1 = INIT$.

Then by Proposition 17(b) replacing q_1 by $COINIT$, still gives us that Q is not a pseudo-subsequence of Q'. Thus, we can use Case 2.

Case 3.2: $q'_1 = COINIT$.

Then by Proposition 17(c) replacing q_1 by $INIT$, still gives us that Q is not a pseudo-subsequence of Q'. Thus, we can use Case 1.

Case 3.3: $q'_1 = HALF$.

In this case, let $i \in R_1$ and consider $\Theta(L^Q_{i,\ldots})$, for some value of other parameters. Suppose it is $L^{Q'}_{j,\ldots}$, for some value of other parameters. Then for all $i' < i$, $i' \in R_1$, $\Theta(L^Q_{i',\ldots})$ must be of form $L^{Q'}_{j',\ldots}$, where $j' \leq j$. Thus, one could essentially consider this as a reduction from $\mathcal{L}^{QQ,RR}$ to $\mathcal{L}^{QQ',RR'}$, where QQ is obtained from Q by replacing q_1 by $INIT$, RR is obtained from R by replacing R_1 by $\{x \in N \mid -x + i \in R_1 - \{y \mid y \geq i\}\}$, QQ' is obtained from Q' by replacing q'_1 with $INIT$, and RR' is obtained from R' by replacing R'_1 by $\{x \in N \mid -x+j \in R'_1 - \{y \mid y \geq j\}\}$. Now we can use Case 1.

It follows from above cases that $\mathcal{L}^{Q,R} \not\leq^{\mathbf{TxtEx}} \mathcal{L}^{Q',R'}$. ∎

Corollary 1 *Suppose $Q \in BASIC^k$ and $Q' \in BASIC^l$. Then, $\mathcal{L}^{Q} \leq^{\mathbf{TxtEx}} \mathcal{L}^{Q'}$ iff Q is a pseudo-subsequence of Q'.*

6 Definitions for Open Semi-Hull and some Propositions

rat denotes the set of non-negative rationals. $\mathbf{rat}^+ = \mathbf{rat} - \{0\}$, denotes the set of positive rationals.

Any language $SEMI\_HULL^n_{\ldots}$ defined below is a geometrical figure "semi-hull," a collection of points in the first quadrant of the plane bounded by the y-axis and a "broken-line" that consists of a straight fragment l_0 of the x-axis (starting from origin) followed by a straight fragment l_1 that makes an angle $\delta_1 < 90°$ with the x-axis, followed by a fragment l_2 that makes an angle $\delta_2 > \delta_1$ with the x-axis, etc. (In above the angle is being measured anti-clockwise from the positive x-axis).

Definition 13 Suppose $a_1, \ldots, a_n \in N$ and $b_1, \ldots, b_n \in \mathbf{rat}^+$, where $0 < a_1 < a_2 < \ldots < a_n$.
$\mathit{SEMI\_HULL}^n_{a_1,b_1,a_2,b_2,\ldots,a_n,b_n} = \{(x,y) \in N^2 \mid y \geq \sum_{1\leq i\leq n} b_i * (x \dot{-} a_i)\}$.

Note that $\mathit{SEMI\_HULL}^0 = N^2$. Also, note that though $\mathit{SEMI\_HULL}^n$ above are subsets of N^2, one can easily consider them as languages $\subseteq N$, by using pairing function. We assume such implicit coding whenever we are dealing with sets $\subseteq N^2$.

Parameters a_i in the above definition specify x-coordinates of "break" points of the border line,while the b_i specify the "slopes" that are being added to the "slope" of the border line after every "break" point.

To make our classes of languages learnable, we have to impose certain restrictions on the parameters a_i, b_i. First, we want both coordinates a and c of "break" points (a,c) to be integers. Secondly, for all languages in our classes, we fix a subset S from which "slopes" b_i may come from. (In the following definition S may be an arbitrary subset of $\mathbf{rat}^+$; however, later we will impose additional restrictions on S). The definition of *valid* sequences of parameters a_i, b_i accomplishes this goal.

Definition 14 Suppose $a_1, \ldots, a_n \in N$ and $b_1, \ldots, b_n \in \mathbf{rat}^+$, where $0 < a_1 < a_2 < \ldots < a_n$. Suppose $S \subseteq \mathbf{rat}^+$.

We say that $(a_1, b_1, \ldots, a_n, b_n)$ is *valid* iff for $1 \leq j \leq n$, $[\sum_{1\leq i\leq n} b_i * (a_j \dot{-} a_i)] \in N$. Additionally, if each $b_i \in S$, then we say that $(a_1, b_1, \ldots, a_n, b_n)$ is *S-valid.*

Let $\mathit{VALID} = \{(a_1, b_1, \ldots, a_n, b_n) \mid (a_1, b_1, \ldots, a_n, b_n) \text{ is valid}\}$.

Let $\mathit{VALID}_S = \{(a_1, b_1, \ldots, a_n, b_n) \mid (a_1, b_1, \ldots, a_n, b_n) \text{ is } S\text{-valid}\}$.

Note that empty sequence () is both valid and S-valid.

Note also that we require $a_1 > 0$. This is for technical convenience, and crucial for some of our results.

Now we define the class of languages we are going to explore.

Definition 15 Suppose $S \subseteq \mathbf{rat}^+$. Then $\mathit{SEMI\_HULL}^{n,S} = \{\mathit{SEMI\_HULL}^n_{a_1,b_1,\ldots,a_n,b_n} \mid (a_1, b_1, \ldots, a_n, b_n) \in \mathit{VALID}_S\}$.

Now we formulate and prove a number of useful technical propositions (few of them immediately follow from the relevant definitions).

Proposition 18 *Suppose* $(a_1, b_1, \ldots, a_j, b_j, a_{j+1}, b_{j+1}) \in \mathit{VALID}$. *Then* $\mathit{SEMI\_HULL}^{j+1}_{a_1,b_1,\ldots,a_j,b_j,a_{j+1},b_{j+1}} \subset \mathit{SEMI\_HULL}^j_{a_1,b_1,\ldots,a_j,b_j}$.

PROOF. Follows from definitions. ∎

Proposition 19 *Suppose* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$ *and* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \in \mathit{VALID}$*, and* $a_j \leq a'_j$*,* $b'_j \geq b_j$*. Then,* $\mathit{SEMI\_HULL}^j_{a_1,b_1,\ldots,a_j,b_j} \subseteq \mathit{SEMI\_HULL}^j_{a_1,b_1,\ldots,a'_j,b'_j}$.

PROOF. Follows from definitions. ∎

Definition 16 Suppose $a_1, b_1, \ldots, , a_j, b_j$ are given such that $(a_1, b_1, \ldots, a_j, b_j) \in \mathit{VALID}$. Then, let $\mathit{INTER}(a_1, b_1, \ldots, a_j, b_j) = \bigcap\{\mathit{SEMI\_HULL}^n_{a_1,b_1,\ldots,a_j,b_j,\ldots,a_n,b_n} \mid n \geq j \wedge (a_1, b_1, \ldots, a_j, b_j, \ldots, a_n, b_n) \in \mathit{VALID}\ \}$.

Intuitively,
$INTER(a_1, b_1, \ldots, a_j, b_j)$, denotes the common portion of all $SEMI\_HULL^n_{a'_1,b'_1,\ldots,a'_j,b'_j,\ldots}$, with $a_i = a'_i$ and $b_i = b'_i$ for $1 \leq i \leq j$.

Proposition 20 *Suppose* $1 \leq j \leq n$, *and* $(a_1, b_1, \ldots, a_j, b_j, \ldots, a_n, b_n) \in VALID$. *Then* $INTER(a_1, b_1, \ldots, a_j, b_j) \subseteq SEMI\_HULL^n_{a_1,b_1,\ldots,a_j,b_j,\ldots,a_n,b_n}$.

Proof. Follows from definitions. ∎

Proposition 21 *Suppose* $(a_1, b_1, \ldots, a_j, b_j, a_{j+1}, b_{j+1})$ *is valid. Then,* $INTER(a_1, b_1, \ldots, a_j, b_j) \subseteq INTER(a_1, b_1, \ldots, a_j, b_j, a_{j+1}, b_{j+1})$.

Proof. Follows from definitions. ∎

Definition 17 Suppose $(a_1, b_1, \ldots, a_j, b_j) \in VALID$. Then let $maxinter(a_1, b_1, \ldots, a_j, b_j)$ denote the least natural number $x > a_j$ such that $\sum_{1 \leq i \leq j} b_i * (x \dot{-} a_i) \in N$.

Proposition 22 *Suppose* $a_1, b_1, \ldots, , a_j, b_j$ *are given such that* $(a_1, b_1, \ldots, a_j, b_j) \in VALID$. *Then,* $INTER(a_1, b_1, \ldots, a_j, b_j) = SEMI\_HULL^n_{a_1,b_1,\ldots,a_j,b_j} \cap \{(x, y) \in N^2 \mid x \leq maxinter(a_1, b_1, \ldots, a_j, b_j)\}$.

Proof. Follows from definitions. ∎

Proposition 23 *Suppose* $a_1, b_1, \ldots, a_{j-1}, b_{j-1}$, a_j, b_j, a'_j, b'_j *are given, where* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$, *and* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ *are valid. If* $INTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$

Then,

(i) $a'_j \leq a_j$ *and*

(ii) $b'_j * (maxinter(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \dot{-} a'_j) \geq b_j * (maxinter(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \dot{-} a_j)$.

Proof. Suppose that $a_1, b_1, \ldots, a_{j-1}, b_{j-1}$, a'_j, b'_j, a_j, b_j are as given in the hypothesis.

If $a_j < a'_j$, then $(a'_j, \sum_{1 \leq i < j} b_i * (a'_j \dot{-} a_i))$ belongs to $INTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ but does not belong to $SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$ (since $(a'_j, \sum_{1 \leq i < j} b_i * (a'_j \dot{-} a_i))$ is "below" the last fragment of the border line for $SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$).

Thus, we must have $a'_j \leq a_j$.

Let $x = maxinter(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$. Let $y = b'_j * (x \dot{-} a'_j) + \sum_{1 \leq i < j} b_i * (x \dot{-} a_i))$. Now, (x, y) belongs to $INTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$. If (x, y) belongs to $SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$ then by definition of $SEMI\_HULL^j$, we must have $y = b'_j * (x \dot{-} a'_j) + \sum_{1 \leq i < j} b_i * (x \dot{-} a_i) \geq b_j * (x \dot{-} a_j) + \sum_{1 \leq i < j} b_i * (x \dot{-} a_i)$. Thus, $b'_j(x \dot{-} a'_j) \geq b_j(x \dot{-} a_j)$. The Proposition Follows. ∎

The following two corollaries have obvious geometrical interpretation.

Corollary 2 *Suppose* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j, \ldots, a'_n, b'_n)$ *and* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a''_j, b''_j, \ldots, a''_n, b''_n)$ *are valid.*

(a) If $a'_j < a''_j$, *then* $SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a''_j,b''_j,\ldots,a''_n,b''_n} \not\subseteq SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a''_j,b''_j,\ldots,a''_n,b''_n}$.

(b) If $a'_j = a''_j$ *and* $b'_j > b''_j$ *then* $SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a''_j,b''_j,\ldots,a''_n,b''_n} \not\subseteq SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j,\ldots,a'_n,b'_n}$.

PROOF. If $SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a''_j,b''_j,\ldots,a''_n,b''_n} \subseteq SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j,\ldots,a'_n,b'_n}$, then it follows that $INTER(a_1,b_1,\ldots,a_{j-1},b_{j-1},a''_j,b''_j) \subseteq SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j}$. Thus it follows from Proposition 23 that $a''_j \leq a'_j$ and if $a''_j = a'_j$, then $b''_j \geq b'_j$. Corollary follows. ∎

Corollary 3 *Suppose* $(a_1,b_1,\ldots,a_n,b_n)$ *and* $(a'_1,b'_1,\ldots,a'_n,b'_n)$ *are valid. Suppose* $SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} \subset SEMI\_HULL^n_{a'_1,b'_1,\ldots,a'_n,b'_n}$. *Let* i *be the minimum value such that* $a_i \neq a'_i$ *or* $b_i \neq b'_i$. *Then,* $a_i < a'_i$ *or* $[a_i = a'_i$ *and* $b_i > b'_i]$.

PROOF. Note that for least j such that $(a'_j,b'_j) \neq (a_j,b_j)$, we must have $INTER^n_{a_1,b_1,\ldots,a_j,b_j} \subseteq SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j}$. Corollary now follows from Proposition 23. ∎

Proposition 24 *Suppose* $1 \leq j \leq n$. *Suppose* $(a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j,\ldots,a_n,b_n)$ *and* $(a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j)$ *are valid.*

(a) If $a'_j \leq a_j$ *and* $b'_j \geq \sum_{j\leq i\leq n} b_i$, *Then,* $SEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j} \subseteq SEMI\_HULL^n_{a_1,b_1,\ldots,a_j,b_j,\ldots,a_n,b_n}$.

(b) If $a'_j \leq a_j$ *and* $b'_j > \sum_{j\leq i\leq n} b_i$, *Then,* $SEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j} \subset SEMI\_HULL^n_{a_1,b_1,\ldots,a_j,b_j,\ldots,a_n,b_n}$.

PROOF. (a) $SEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j} = \{(x,y) \in N^2 \mid y \geq b'_j(x \dot{-} a'_j) + \sum_{1\leq i<j} b_i * (x \dot{-} a_i)\}$.

$SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} = \{(x,y) \in N^2 \mid y \geq \sum_{j\leq i\leq n} b_i * (x \dot{-} a_i) + \sum_{1\leq i<j} b_i * (x \dot{-} a_i)\}$.

Part (a) follows since $b'_j(x \dot{-} a'_j) \geq b'_j(x \dot{-} a_j) \geq \sum_{j\leq i\leq n} b_i * (x \dot{-} a_j) \geq \sum_{j\leq i\leq n} b_i * (x \dot{-} a_i)$.

(b) Follows from part (a) and Corollary 2. ∎

Now we are at the point when our results require additional constraint on the set S (of "slopes"). Intuitively, the set S satisfying the constraints "cover" the positive rational numbers, and can be algorithmically listed in a monotonic order on a two-sided-infinite tape. A natural example of set S satisfying the constraint below is the set $N \bigcup \{1/x | x \in N\}$. Although our results below hold for any fixed set S of rationals satisfying the constraint in question, we suggest the reader to keep in mind the above set when reading the proofs.

Definition 18 A set $S \subseteq \mathbf{rat}^+$ is said to be $\mathbf{rat}^+$-*covering* iff there exists a recursive bijection f from Z to S such that,

(i) for $i, j \in Z$, $i < j$ iff $f(i) < f(j)$.

(ii) for every $x \in \mathbf{rat}^+$, there exist $y, y' \in S$ such that $y < x < y'$.

(A natural choice for a set S (which doesn't satisfy the above constraint) seems to be the set $\mathbf{rat}^+$. However, in this case, a complete class of languages $\{L_y = \{x | x \in \mathbf{rat}^+, x \geq y\} \mid y \in \mathbf{rat}^+\}$ (see [JKW99]) would be trivially reducible to any class of languages-figures considered in our paper, thus making all of them of the same complexity. The use of $\mathbf{rat}^+$-*covering* sets S gives us opportunity to capture differences in learnability of different geometrical concepts observed in our paper).

Our results below hold for any $\mathbf{rat}^+$-*covering* set S. However it is open at present whether $SEMI\_HULL^{n,S} \equiv^{\mathbf{TxtEx}} SEMI\_HULL^{n,S'}$, for arbitrary $\mathbf{rat}^+$-*covering* sets S and S'.

Note that, for any given $n \in N$, and $\mathbf{rat}^+$-*covering* set S, there exists a limiting partial recursive function F such that, if i is a grammar for some $SEMI\_HULL^{n,S}_{a_1,b_1,\ldots,a_n,b_n}$, where

$(a_1, b_1, \ldots, a_n, b_n)$ being S-valid, then $F(i)$ converges to $(a_1, b_1, \ldots, a_n, b_n)$. That is one can determine the parameters limit-effectively from any grammar for $\mathit{SEMI\_HULL}^{n,S}_{a_1,b_1,\ldots,a_n,b_n}$. This fact would be implicitly used in construction of Ψ in some of our proofs.

Intuitively, the following technical proposition claims that, given a point (x_0, y_0) in a language $\mathit{SEMI\_HULL}^{j-1}_{a_1,b_1,\ldots,a_{j-1},b_{j-1}}$, one can effectively find an lower bound on the boundary "slopes" b_j of all languages $\mathit{SEMI\_HULL}^{j}_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j,\ldots}$ which do not contain (x_0, y_0).

Proposition 25 *Suppose $(a_1, b_1, \ldots, a_{j-1}, b_{j-1})$ is valid. Let $(x_0, y_0) \in N^2$ be such that $y_0 > \sum_{1\le i<j} b_i(x_0 \dot{-} a_i)$. Then, there exists a $B' \in$ **rat**$^+$ obtainable effectively from $a_1, b_1, \ldots, a_{j-1}, b_{j-1}, x_0$ and y_0 such that for any (a'_j, b'_j), if $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$, is valid and $x_0 \le a'_j$ or $b'_j < B'$, then, $(x_0, y_0) \in \mathit{INTER}^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j}$.*

Proof. If $a'_j \ge x_0$, then we would have $y_0 \ge \sum_{1\le i<j} b_i(x_0 \dot{-} a_i) = b'_j(x_0 \dot{-} a'_j) + \sum_{1\le i<j} b_i(x_0 \dot{-} a_i)$. Thus, $(x_0, y_0) \in \mathit{INTER}^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j}$.

Now fix any $a'_j < x_0$. If $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ is valid, but $(x_0, y_0) \notin \mathit{INTER}^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j}$, then either (a) $\mathit{maxinter}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) < x_0$, or (b) $y_0 < b'_j(x_0 \dot{-} a'_j) + \sum_{1\le i<j} b_i(x_0 \dot{-} a_i)$.

Let $B(a'_j) = (y_0 - \sum_{1\le i<j} b_i(x_0 \dot{-} a_i))/(x_0 \dot{-} a'_j)$. Now for (b) to be true b'_j must be greater than $B(a'_j)$. Then, for any $b'_j \le B(a'_j)$ to satisfy (a), for $x_1 = \mathit{maxinter}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$, and $y_1 = b'_j * (x_1 \dot{-} a'_j) + \sum_{1\le i<j} b_i * (x_1 \dot{-} a_i)$, (x_1, y_1) must lie in the intersection of the following three regions: (A) $y > \sum_{1\le i<j} b_i(x \dot{-} a_i)$, (B) $x \le x_0$, and (C) $y \le B(a'_j) * (x \dot{-} a'_j) + \sum_{1\le i<j} b_i(x \dot{-} a_i)$.

Since the above region is bounded, there are only finitely many possibilities for (x_1, y_1), and thus for b'_j. Thus, let $B_1(a'_j) \le B(a'_j)$, be a positive rational number, such that $\mathit{maxinter}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) < x_0$ implies $b'_j > B_1(a'_j)$. Note that such a $B_1(a'_j)$ exists and can be obtained effectively from $a_1, b_1, \ldots, a_{j-1}, b_{j-1}, x_0, y_0$.

Now taking $B' = \min(\{B_1(a'_j) \mid a_{j-1} < a'_j < x_0\})$, witnesses the Proposition. ∎

Corollary 4 *Suppose $(a_1, b_1, \ldots, a_j, b_j)$ is valid. Then, one can effectively (in $a_1, b_1, \ldots, a_j, b_j$) obtain a $B' \in$ **rat**$^+$ such that following property is satisfied.*

For any $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ which is valid, if $\mathit{INTER}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq \mathit{SEMI\_HULL}^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$, then $b'_j \ge B'$ and $a'_j \le a_j$.

Proof. Pick $(x_0, y_0) \in N^2$ such that $x_0 > a_j$, and $\sum_{1\le i<j} b_i(x_0 \dot{-} a_i) < y_0 < \sum_{1\le i\le j} b_i(x_0 \dot{-} a_i)$. Corollary, now follows from Proposition 23 and 25. ∎

Proposition 27 below will play an important role in some of our proofs. Imagine that, in the process of learning, $\mathit{SEMI\_HULL}^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j,\ldots,a_n,b_n}$, the learner has already learned the parameters $a_1, b_1, \ldots, a_{j-1}, b_{j-1}$, and is now trying to learn the parameters a_j, b_j. Our aim is to use a $(\mathit{INIT}, \mathit{COINIT})$ type strategy to learn (a_j, b_j) (instead of the trivial $(\mathit{INIT}, \mathit{HALF})$ type strategy mentioned in the introduction).

Since we intend to use $(\mathit{INIT}, \mathit{COINIT})$ type strategy, we need to be "safe" in choosing the parameters. That is, choosing (a_j, b_j) must imply that any (a'_j, b'_j) "smaller" than (a_j, b_j) must be inconsistent with the input, ("smaller" in the sense of some $(\mathit{INIT}, \mathit{COINIT})$-type ordering of all pairs (a''_j, b''_j)).

This is achieved in two steps. First we order (a''_j, b''_j), with $b''_j \le B$, for some fixed positive rational number B in a INIT like fashion. Then, we order $b''_j > B$, in a COINIT like fashion. This latter ordering is easily achievable (as will be seen in the proof of Theorem 3 below). For the former ordering, if $b'_j, b_j \leq B$, then we make sure that if $\mathit{INTER}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ is a subset of $\mathit{SEMI\_HULL}^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$, then ordering places (a'_j, b'_j) below (a_j, b_j). Proposition 27 (which is based on the Corollary 5 and Proposition 26 below) shows that this is achievable. Note that this ordering depends on $(a_1, b_1, \ldots, a_{j-1}, b_{j-1})$.

Corollary 5 *Suppose $1 \leq j \leq n$. Let S be any* **rat**$^+$*-covering set, and $B \in S$.*

Suppose $a_1, b_1, \ldots, a_{j-1}, b_{j-1}$ a_j, b_j are given, where $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$ is S-valid. Then, there exist only finitely many (a'_j, b'_j), such that

(i) $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ is S-valid,

(ii) $b'_j \leq B$, and

(iii) $\mathit{INTER}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq \mathit{SEMI\_HULL}^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$.

Moreover, canonical index for the finite set of a'_j, b'_j satisfying above three conditions can be obtained effectively from $B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j$.

Furthermore, for any a'_j, b'_j satisfying the above three conditions, $a'_j \leq a_j$, and if $a'_j = a_j$, then $b'_j \geq b_j$.

Proof. Furthermore clause follows from Proposition 23.

By Corollary 4, one can effectively find a $B' \in$ **rat**$^+$ such that any (a'_j, b'_j) satisfying (i) and (iii) must satisfy $b'_j \geq B'$ and $a'_j \leq a_j$. Corollary now follows since there are only finitely many (a'_j, b'_j) such that $a'_j \leq a_j$, $B' \leq b'_j \leq B$, and $b'_j \in S$ and for any (a'_j, b'_j), one can effectively test whether (a'_j, b'_j) satisfies clauses (i) to (iii) in the corollary. ∎

The above corollary together with the following technical proposition will enable us to impose a suitable ordering on $(a_1, b_1, \ldots, a_j, b_j)$ with an upper bound B on "slopes" b_j. Think of $(a_1, b_1, \ldots, a_{j-1}, b_{j-1})$ as "already fixed."

Proposition 26 *Suppose R is a partial order over an r.e. set A, and F is a partial recursive function with domain A such that, for any $x \in A$,*

(i) $\{x' \mid x'Rx\}$ is finite, and

(ii) $F(x)$ is the canonical index for $\{x' \in A \mid x'Rx\}$.

Then, effectively from a program for F, one can obtain a program for a 1–1, partial recursive function h with domain A and range $\subseteq N$, such that $x'Rx$ implies $h(x') \leq h(x)$. Moreover, if A is infinite, then range of h is N.

Proof. Let f be a recursive function such that range$(f) = A$. Define h as follows. Let $cur = 0$. Go to stage 0.

Stage s

(* In this stage we make sure that $h(f(s))$, along with $h(x)$, for all x such that $xRf(s)$, are defined appropriately. *)

If $h(f(s))$ has already been defined, then go to stage $s+1$.

Otherwise let $X = \{x \mid xRf(s)\}$ (note that one can effectively obtain X from $f(s)$ by assumption (ii) of the hypothesis; also note that $f(s) \in X$, since $f(s)Rf(s)$).

Let $x_0, x_1, \ldots, x_m$ be listing of members of X such that x_iRx_j implies $i \leq j$ (note that one can easily get such an ordering since X is finite).

For $j = 0$ to m do

If $h(x_j)$ has not been defined then

let $h(x_j) = cur$; $cur = cur + 1$.

EndFor

(* note that $h(f(s))$ along with $h(x)$, for all x such that $xRf(s)$, have been defined. *)

End Stage s

It is easy to verify that h satisfies the properties required in the proposition. ∎

Proposition 27 *Let S be any* $\mathbf{rat}^{+}$*-covering set. Then, there exists a recursive function* code *with domain* $\mathbf{rat} \times VALID_S$*, and range* $\subseteq N$ *such that following is satisfied.*

Suppose $1 \leq j \leq n$, $B \in S$, *and* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1})$ *is* S*-valid.*

(A) Suppose $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$ *and* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ *are* S*-valid.*

(A.1) If $b_j, b'_j \leq B$ *and* $INTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq SEMI\_HULL^{j}_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$, *then* $\mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \leq \mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$.

(A.2) If $b_j, b'_j \leq B$ *and* $(a_j, b_j) \neq (a'_j, b'_j)$, *then* $\mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \neq \mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$.

(A.3) $\{\mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a''_j, b''_j) \mid (a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a''_j, b''_j) \in VALID_S$ *and* $b''_j \leq B\} = N$.

(B) Suppose $(a_1, b_1, \ldots, a_j, b_j)$ *is* S*-valid. If* $b_j > B$, *then* $\mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j) = \mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, B)$.

Proof. Fix $a_1, b_1, \ldots, a_{j-1}, b_{j-1}$ and B as in the hypothesis. Let us first define a relation Rel on $N \times S \cap \{r \in \mathbf{rat} \mid r \leq B\}$ as follows:

$(a', b')\mathrm{Rel}(a, b)$ iff

$INTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a', b') \subseteq SEMI\_HULL^{j}_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a,b}$.

Note that (by Corollary 5) for each (a, b), there are only finitely many (a', b') such that $(a', b')\mathrm{Rel}(a, b)$. Also, for each of these (a', b'), $a' \leq a$ and either $a' < a$ or $b' \geq b$. Furthermore, canonical index for $\{(a', b') \mid (a', b')\mathrm{Rel}(a, b)\}$, can be effectively obtained from (a, b).

Let Rel^* be transitive closure of Rel. It is easy to verify that Rel^* is a partial order, where for each (a, b), there exists at most finitely many (a', b') such that $(a', b')\mathrm{Rel}^*(a, b)$. Moreover, canonical index for $\{(a', b') \mid (a', b')\mathrm{Rel}^*(a, b)\}$, can be effectively obtained from (a, b).

Existence of code as required now follows from Proposition 26. ∎

7 Q-classes to Which $SEMI\_HULL^{n,S}$ is Reducible

Our goal in this section is to establish an upper bound on the $SEMI\_HULL^{n,S}$ degrees in terms of Q-hierarchy. To find such a bound, we actually have to design a learning strategy for languages in $SEMI\_HULL^{n,S}$ that consists of q_i-type strategies for some $Q = (q_1, q_1, \ldots, q_k)$, and a grammar learned by every q_i is used by q_{i+1}. A natural strategy of this type would be the following $(q_1, q_2, \ldots, q_{2n-1}, q_{2n})$-strategy, where $q_{2i+1} = INIT$ and $q_{2i+2} = HALF$ for $i < n$: learn the first "break" point a_1 using an $INIT$-type strategy; once a_1 has been learned, learn the first "slope" b_1 at the point $(a_1, 0)$ using a $HALF$-type strategy; then learn the second "break" point $(a_2, b_1 * (a_2 - a_1))$ using an $INIT$-type strategy, etc. However, a much more "efficient" learning strategy is suggested by the following.

Theorem 3 *Suppose S is* $\mathbf{rat}^{+}$*-covering. Suppose* $Q = (q_1, q_2, \ldots, q_{2n-1}, q_{2n})$, *where* $q_{2i+1} = INIT$ *and* $q_{2i+2} = COINIT$, *for* $i < n$. *Then* $SEMI\_HULL^{n,S} \leq^{\mathbf{TxtEx}} \mathcal{L}^{Q}$.

PROOF. Intuitive idea of the learning strategy is as follows. Suppose we have already learned $(a_1, b_1, \ldots, a_{j-1}, b_{j-1})$. Then, we use *INIT* like stratagy to learn any pair (a_j, b_j), if $b_j \leq B = h(0)$, and use *COINIT* type strategy to learn b_j if $b_j > B$. The former is done using code as in Proposition 27. The latter can be done easily by using h to form a *COINIT* like strategy.

Going through the proof below (and the proofs for other upper bounds in this paper) one must be aware of the fact that, though intuitively we do use *INIT*/*COINIT* (or whatever - in other proofs) strategy for learning each successive parameter, in the actual proof we do it by choosing "least consistent" *SEMI_HULL* on every step and *INIT*/*COINIT* strategy is in some sense hidden.

Now we proceed with the formal construction.

Fix S which is **rat**$^+$*-covering*. Let h be a recursive bijection from Z to S such that $h(i) < h(i+1)$. Let $B = h(0)$.

Fix code as in Proposition 27. Let g be a function from S to S such that $g(b) = 0$, if $b \leq B$, and $g(b) = h^{-1}(b)$, otherwise. (Note that g is monotonically non-decreasing).

The following assertion utilizes the construction of proposition 27 to show that the values $(code(B, a_1, b_1, \ldots, a_j, b_j))$ can be used as conjectures by *INIT*-type "substrategies", and the values $g(b_j)$ can be used as conjectures by *COINIT*-type "substrategies".

Claim 1 *Suppose* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j), (a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \in VALID_S$, *where* $(a_j, b_j) \neq (a'_j, b'_j)$. *Suppose* $INTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j) \subset SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j}$.

Then,

(i) $\mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j) < \mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$, *or*

(ii) $\mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j) = \mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$, *and* $g(b_j) > g(b'_j)$.

PROOF. By Proposition 23 we have that $a_j \leq a'_j$.

We now consider following cases:

Case 1: $b_j, b'_j \leq B$.

Since, $INTER(a_1, b_1, \ldots, a_j, b_j) \subseteq SEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j}$, we have $\mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j) < \mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ (by Proposition 27; for getting $<$ instead of $\leq$ use the fact that $(a_j, b_j) \neq (a'_j, b'_j)$).

Case 2: $b_j \leq B < b'_j$.

In this case $a_j < a'_j$ by Proposition 23.

Also, $INTER(a_1, b_1, \ldots, a_j, b_j) \subseteq SEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j} \subseteq SEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,B}$.

Thus, $\mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j) < \mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, B) = \mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ (by Proposition 27; for getting $<$ instead of $\leq$ use the fact that $(a_j, b_j) \neq (a'_j, B)$).

Case 3: $b'_j \leq B < b_j$.

In this case, $INTER(a_1, b_1, \ldots, a_j, B) \subseteq SEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,B} \subseteq SEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j}$. Thus, $\mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j) = \mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, B) \leq \mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ (by Proposition 27). Also $g(b'_j) = 0 < g(b_j)$.

Case 4: $B < b_j \leq b'_j$.

In this case $a_j < a'_j$ by Proposition 23. Thus, $INTER(a_1, b_1, \ldots, a_j, B) \subseteq SEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,B} \subseteq SEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,B}$.

Thus, $\text{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j) = \text{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, B) < \text{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, B) = \text{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ (by Proposition 27; for getting $<$ instead of $\leq$ use the fact that $(a_j, B) \neq (a'_j, B)$).

Case 5: $B < b'_j \leq b_j$.

If $a_j < a'_j$, then $INTER(a_1, b_1, \ldots, a_j, B) \subseteq SEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,B} \subseteq SEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,B}$. Thus, $\text{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j) = \text{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, B) < \text{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, B) = \text{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ (by Proposition 27; for getting $<$ instead of $\leq$ use the fact that $(a_j, B) \neq (a'_j, B)$).

If $a_j = a'_j$, then $b'_j < b_j$ (by Proposition 23). Thus, $g(b_j) > g(b'_j)$. Also, $INTER(a_1, b_1, \ldots, a_j, B) \subseteq SEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,B}$. Thus, $\text{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j) = \text{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, B) \leq \text{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, B) = \text{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ (by Proposition 27). $\square$

Let map be a mapping from $VALID$ to N^* such that

$map(a_1, b_1, \ldots, a_n, b_n) = (\text{code}(B, a_1, b_1), g(b_1), \text{code}(B, a_1, b_1, a_2, b_2), g(b_2), \ldots, \text{code}(B, a_1, b_1, \ldots, a_n, b_n), g(b_n))$.

Now we suggest the reader to recall the properties of the ordering $<_Q$. For example, note that if $Q = (INIT, COINIT)$ then $(i, j) <_Q (i', j')$ would mean that $i < i'$ or ($i = i'$ and $j > j'$).

Claim 2 *Suppose* $(a_1, b_1, \ldots, a_n, b_n)$ *and* $(a'_1, b'_1, \ldots, a'_n, b'_n)$ *are S-valid.*

(A) If $map(a'_1, b'_1, \ldots, a'_n, b'_n) <_Q map(a_1, b_1, \ldots, a_n, b_n)$*, then, for the least* j *such that* $(a_j, b_j) \neq (a'_j, b'_j)$*,* $INTER(a_1, b_1, \ldots, a_j, b_j) \not\subseteq SEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j}$.

(B) If $map(a'_1, b'_1, \ldots, a'_n, b'_n) <_Q map(a_1, b_1, \ldots, a_n, b_n)$*, Then,* $INTER(a_1, b_1, \ldots, a_j, b_j, \ldots, a_n, b_n) \not\subseteq SEMI\_HULL^n_{a'_1,b'_1,\ldots,a'_n,b'_n}$.

(C) If $SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} \subset SEMI\_HULL^n_{a'_1,b'_1,\ldots,a'_n,b'_n}$*, then* $map(a_1, b_1, \ldots, a_n, b_n) <_Q map(a'_1, b'_1, \ldots, a'_n, b'_n)$*.*

PROOF. (A) Let j be least number such that $(a_j, b_j) \neq (a'_j, b'_j)$. Note that, for $i < j$, we must have $\text{code}(B, a_1, b_1, \ldots, a_i, b_i) = \text{code}(B, a'_1, b'_1, \ldots, a'_i, b'_i)$, and $g(b_i) = g(b'_i)$. If $INTER(a_1, b_1, \ldots, a_j, b_j) \subseteq SEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j}$, then, by Claim 1 we would have $\text{code}(B, a_1, b_1, \ldots, a_j, b_j) < \text{code}(B, a'_1, b'_1, \ldots, a'_j, b'_j)$ or $\text{code}(B, a_1, b_1, \ldots, a_j, b_j) = \text{code}(B, a'_1, b'_1, \ldots, a'_j, b'_j)$ and $g(b_j) > g(b'_j)$. Thus, $map(a_1, b_1, \ldots, a_n, b_n) <_Q map(a'_1, b'_1, \ldots, a'_n, b'_n)$, a contradiction to the hypothesis.

(B) Let j be the least number such that $(a_j, b_j) \neq (a'_j, b'_j)$. Now (B) follows using part (A) and the fact that $INTER(a_1, b_1, \ldots, a_j, b_j) \subseteq INTER(a_1, b_1, \ldots, a_j, b_j, \ldots, a_n, b_n)$ and $SEMI\_HULL^n_{a'_1,b'_1,\ldots,a'_j,b'_j,\ldots,a'_n,b'_n} \subseteq SEMI\_HULL^j_{a'_1,b'_1,\ldots,a'_j,b'_j}$.

(C) Follows from part (B) and $INTER(a_1, b_1, \ldots, a_j, b_j, \ldots, a_n, b_n) \subseteq SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}$. $\square$

We now continue with the proof of the theorem. The aim is to construct Θ which maps $SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}$ to $L^Q_{map(a_1,b_1,\ldots,a_n,b_n)}$.

Note that definition of Ψ mapping grammar sequence converging to a grammar for $L^Q_{map(a_1,b_1,\ldots,a_n,b_n)}$ to a grammar sequence converging to a grammar for $SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}$ would be trivial. We thus just define Θ.

Without loss of generality, we will be giving Θ as mapping sets to sets.

For any finite $X \subseteq N^2$, let $Prop(X, a_1, b_1, \ldots, a_n, b_n)$ be true iff following two properties are satisfied.

(A) $(a_1, b_1, \ldots, a_n, b_n) \in \mathit{VALID}_S$.

(B) For all $(a'_1, b'_1, \ldots, a'_n, b'_n) \in \mathit{VALID}_S$ such that $map(a'_1, b'_1, \ldots, a'_n, b'_n) <_Q map(a_1, b_1, \ldots, a_n, b_n)$, $X \not\subseteq \mathit{SEMI\_HULL}^n_{a'_1, b'_1, \ldots, a'_n, b'_n}$.

Note that condition (B) above is equivalent to

(B') For all j, $1 \le j \le n$, (B'.1) and (B'.2) are satisfied.

(B'.1) For all $a'_j \in N, b'_j \in S$, $b'_j \le B$,

If $\mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) < \mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j))$, then

$[X \not\subseteq \mathit{SEMI\_HULL}^j_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j}]$.

(B'.2) If $B \le b_j$, then for all $b'_j \in S$ such that $b_j < b'_j$, $X \not\subseteq \mathit{SEMI\_HULL}^j_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b'_j}$

Note that B'.2 above is equivalent to

(B".2) If $B \le b_j$, then for least $b'_j \in S$ such that $b_j < b'_j$, $X \not\subseteq \mathit{SEMI\_HULL}^j_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b'_j}$.

Note that whether $X, a_1, b_1, \ldots, a_n, b_n$, satisfy (A) and (B'.1) and (B".2) for all j, $1 \le j \le n$, is effectively testable.

Thus, for any finite set $X \subseteq N^2$, let

$\Theta(X) = \bigcup\{L^Q_{map(a_1, b_1, \ldots, a_n, b_n)} \mid \mathit{Prop}(X, a_1, b_1, \ldots, a_n, b_n)\}$.

For infinite X', $\Theta(X') = \bigcup_{X \subseteq X', \mathrm{card}(X) < \infty} \Theta(X)$.

It is easy to verify that

(1) for any $X \subseteq \mathit{SEMI\_HULL}^n_{a_1, b_1, \ldots, a_n, b_n}$,

$\Theta(X) \subseteq L^Q_{map(a_1, b_1, \ldots, a_n, b_n)}$ (due to clause (B) in definition of Prop above, and the fact that for any valid I and I', $map(I) <_Q map(I')$, implies $L^Q_{map(I)} \subseteq L^Q_{map(I')}$), and

(2) For any finite set $X \subseteq N^2$ such that $\{(x, y) \in N^2 \mid x \le maxinter(a_1, b_1, \ldots, a_n, b_n)$ and $y = \min(\{y' \mid (x, y') \in \mathit{INTER}(a_1, b_1, \ldots, a_n, b_n)\})\} \subseteq X \subseteq \mathit{SEMI\_HULL}^n_{a_1, b_1, \ldots, a_n, b_n}$,

$\Theta(X) \supseteq L^Q_{map(a_1, b_1, \ldots, a_n, b_n)}$.

(By Claim 2(B), and definition of Prop and Θ).

Thus, we have that $\Theta(\mathit{SEMI\_HULL}^n_{a_1, b_1, \ldots, a_n, b_n}) = L^Q_{map(a_1, b_1, \ldots, a_n, b_n)}$. ∎

Now we will show that the above theorem is in some sense optimal. That is, for $Q = (q_1, q_2, \ldots, q_{2n-1}, q_{2n})$, where $q_{2i+1} = \mathit{INIT}$, and $q_{2i+2} = \mathit{COINIT}$, for $i < n$, and any $Q' \in \mathit{BASIC}^*$, if $\mathcal{L}^Q \not\le^{\mathbf{TxtEx}} \mathcal{L}^{Q'}$, then $\mathit{SEMI\_HULL}^{n,S} \not\le^{\mathbf{TxtEx}} \mathcal{L}^{Q'}$. Thus, Q in above theorem cannot be improved if we use components only from BASIC (whether we can improve it by using some other basic components is open).

Theorem 4 *Suppose $n \in N^+$, $1 \le j \le n$. Suppose S is any* $\mathbf{rat}^+$*-covering set.*

(1) Suppose $(a_1, b_1, \ldots, a_{j-1}, b_{j-1})$ is S-valid.

Let $Q = (q_1, q_2, \ldots, q_{2(n-j+1)})$, where $q_{2i+1} = \mathit{INIT}$ and $q_{2i+2} = \mathit{COINIT}$ (for $i \le n-j$). Suppose $Q' \in \mathit{BASIC}^*$ is such that $\mathcal{L}^Q \not\le \mathcal{L}^{Q'}$.

Then $\{\mathit{SEMI\_HULL}^n_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j \ldots, a'_n, b'_n} \mid (a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j \ldots, a'_n, b'_n)$ is S-valid $\} \not\le^{\mathbf{TxtEx}} \mathcal{L}^{Q'}$.

(2) Suppose $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j)$ is such that there exists a b_j, such that $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$ is S-valid.

Let $Q = (q_1, \ldots, q_{2(n-j)+1})$, where $q_{2i+1} = \mathit{COINIT}$, for $i \le n-j$, and $q_{2i+2} = \mathit{INIT}$, for $i < n-j$. Suppose $Q' \in \mathit{BASIC}^*$ is such that $\mathcal{L}^Q \not\le \mathcal{L}^{Q'}$.

Then $\{\mathit{SEMI\_HULL}^n_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b'_j \ldots, a'_n, b'_n} \mid (a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b'_j \ldots, a'_n, b'_n)$ is S-valid $\} \not\le^{\mathbf{TxtEx}} \mathcal{L}^{Q'}$.

Intuitive idea of proof is to use a's to diagonalize against sequences of *COINIT* and b's to diagonalize against sequences of *INIT*.

PROOF. For a fixed n, we prove the above theorem by reverse induction on j (from $j = n$ to $j = 1$). For each such j, we first show (2) and then (1).

Base Case: $j = n$, for (2):

In this case $Q = (COINIT)$.

Now $COINIT \leq^{\mathbf{TxtEx}} \{SEMI\_HULL^n_{a_1,b_1,\ldots,a_{n-1},b_{n-1},a_n,b'_n} \mid (a_1, b_1, \ldots, a_{n-1}, b_{n-1}, a_n, b'_n)$ is S-valid $\}$ as witnessed by Θ and Ψ defined as follows.

Let h be a isomorphism from Z to S such that $h(i) < h(i + 1)$.

For any set $Y \subseteq N$, let $\Theta(Y) = \bigcup_{i\in Y} SEMI\_HULL^n_{a_1,b_1,\ldots,a_{n-1},b_{n-1},a_n,h(i)}$.

It is easy to verify that $\Theta(L^Q_i) = SEMI\_HULL^n_{a_1,b_1,\ldots,a_{n-1},b_{n-1},a_n,h(i)}$. Let Ψ be defined as follows. If a sequence α of grammars converges to a grammar for $SEMI\_HULL^n_{a_1,b_1,\ldots,a_{n-1},b_{n-1},a_n,h(i)}$, then $\Psi(\alpha)$ converges to a grammar for L^Q_i.

It is easy to verify that Θ and Ψ witness that $COINIT \leq^{\mathbf{TxtEx}} \{SEMI\_HULL^n_{a_1,b_1,\ldots,a_{n-1},b_{n-1},a_n,b'_n} \mid (a_1, b_1, \ldots, a_{n-1}, b_{n-1}, a_n, b'_n)$ is S-valid $\}$.

Thus, since $COINIT \not\leq^{\mathbf{TxtEx}} \mathcal{L}^{Q'}$, (2) is proven for $j = n$.

Induction Case (for (1)):

Suppose for $j > j'$, we have shown both (1) and (2), and for $j = j'$, we have shown (2). Then we show (1) for $j = j'$.

Note that $Q = (q_1, q_2, \ldots, q_{2(n-j+1)})$, where q_{2i+1} is *INIT* and q_{2i+2} is *COINIT*, for $i \leq n-j$. Without loss of generality assume that Q' contains at least one *INIT* or *HALF* (otherwise just add *INIT* at the end of Q'). Suppose $Q' = (q'_1, \ldots, q'_{k-1}, q'_k, q'_{k+1}, \ldots, q'_l)$, where $q'_1, \ldots, q'_{k-1}$ are *COINIT*, q'_k may be either *INIT* or *HALF*, and for $k+1 \leq i \leq l$, $q'_i \in \{INIT, COINIT, HALF\}$. Without loss of generality assume that q'_k is *INIT* (since otherwise we could just replace $q'_k = HALF$ with $(COINIT, INIT)$ and consider $Q' = (q'_1, \ldots, q'_{k-1}, COINIT, INIT, q'_{k+1}, \ldots, q'_l)$).

Now, by hypothesis we know that $\mathcal{L}^Q \not\leq^{\mathbf{TxtEx}} \mathcal{L}^{Q'}$. Thus, by Corollary 1, Q is not a pseudo-subsequence of Q'. Now consider Q'' obtained from Q by dropping $q_1 = INIT$, Q''' obtained from Q' by dropping $q'_1, q'_2, \ldots, q'_{k-1}$. Now, Q is not a pseudo-subsequence of Q''' by repeated use of Proposition 16(b), and thus Q'' is not a pseudo-subsequence of Q''' by Proposition 17(a). Thus, by Corollary 1, $\mathcal{L}^{Q''} \not\leq^{\mathbf{TxtEx}} \mathcal{L}^{Q'''}$.

Now, suppose by way of contradiction that Θ along with Ψ witnesses the reduction

$\{SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j\ldots,a'_n,b'_n} \mid (a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j \ldots, a'_n, b'_n)$ is S-valid $\}$ $\leq^{\mathbf{TxtEx}} \mathcal{L}^{Q'}$.

Recall that $SEMI\_HULL^0$ is N^2. Let $v_1 = \min(\{u_1 \mid (\exists u_2, \ldots, u_l)[\langle u_1, u_2, \ldots, u_l\rangle \in \Theta(SEMI\_HULL^{j-1}_{a_1,b_1,\ldots,a_{j-1},b_{j-1}})]\})$

For $1 < i < k$, let $v_i = \min(\{u_i \mid (\exists u_{i+1}, \ldots, u_l)[\langle v_1, \ldots, v_{i-1}, u_i, \ldots, u_l\rangle \in \Theta(SEMI\_HULL^{j-1}_{a_1,b_1,\ldots,a_{j-1},b_{j-1}})]\})$.

Let σ be such that content$(\sigma) \subseteq SEMI_HULL^{j-1}_{a_1,b_1,\ldots,a_{j-1},b_{j-1}}$, and there exist $u_k, \ldots, u_l$, such that $\langle v_1, v_2, \ldots, v_{k-1}, u_k, \ldots, u_l\rangle \in \Theta(\sigma)$.

Let $a_j > \max(\{x \in N \mid (\exists y \in N)[\langle x, y\rangle \in \text{content}(\sigma)]\})$ be such that, for some b_j, $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$ is S-valid. Note that, for all $b''_j, a''_{j+1}, b''_{j+1}, \ldots, a''_n, b''_n$, such that $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b''_j, a''_{j+1}, b''_{j+1}, \ldots, a''_n, b''_n)$ is S-valid, content$(\sigma) \subseteq SEMI_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b''_j,a''_{j+1},b''_{j+1},\ldots,a''_n,b''_n}$. Thus, Θ (along with Ψ) essentially witnesses that

$\{SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b'_j\ldots,a'_n,b'_n} \mid (a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b'_j \ldots, a'_n, b'_n)$ is S-valid, $\}$ $\leq^{\mathbf{TxtEx}} \mathcal{L}^{Q'''}$ (since parameters for the first $k-1$ *COINIT*s in Q' are fixed to be $v_1, \ldots, v_{k-1}$).

This is not possible by induction hypothesis (since $\mathcal{L}^{Q''} \not\leq^{\mathbf{TxtEx}} \mathcal{L}^{Q'''}$).

Induction Case (for (2)): Suppose for $j > j'$, we have shown both (1) and (2). Then we show (2) for $j = j'$.

In this case $Q = (q_1, q_2, \ldots, q_{2(n-j)+1})$, where q_{2i+1} is *COINIT*, for $i \leq n-j$, and q_{2i+2} is *INIT*, for $i < n-j$. Without loss of generality assume that Q' contains at least one *COINIT* or *HALF* (otherwise just add *COINIT* to the end of Q'). Suppose Q' is of form $(q'_1, \ldots, q'_{k-1}, q'_k, \ldots, q'_l)$, where $q'_1, \ldots, q'_{k-1}$ are *INIT*, q'_k may be either *COINIT* or *HALF*, and for $k+1 \leq i \leq l$, $q'_i \in \{INIT, COINIT, HALF\}$. Without loss of generality assume that q'_k is *COINIT* (since otherwise we could just replace $q'_k = HALF$ with $(INIT, COINIT)$ and consider $Q' = (q'_1, \ldots, q'_{k-1}, INIT, COINIT, q'_{k+1}, \ldots, q'_l)$).

Now, by hypothesis we know that $\mathcal{L}^{Q} \not\leq^{\mathbf{TxtEx}} \mathcal{L}^{Q'}$. Thus, by Corollary 1, Q is not a pseudo-subsequence of Q'. Thus, for Q'' obtained from Q by dropping $q_1 = COINIT$, Q''' obtained from Q' by dropping $q'_1, q'_2, \ldots, q'_{k-1}$ (note that q'_k is *COINIT*, while the first q in Q'' is *INIT*), Q'' is not a pseudo-subsequence of Q''' (this case is similar to the one in Induction Case for (1) with *INIT*s and *COINIT*s reversed). Thus, by Corollary 1, $\mathcal{L}^{Q''} \not\leq^{\mathbf{TxtEx}} \mathcal{L}^{Q'''}$.

Suppose by way of contradiction that Θ along with Ψ witnesses the reduction

$\{SEMI\_HULL^{n}_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b'_j\ldots,a'_n,b'_n} \mid (a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b'_j \ldots, a'_n, b'_n)$ is S-valid $\}$ $\leq^{\mathbf{TxtEx}} \mathcal{L}^{Q'}$.

Let $X = \{\langle u_1, u_2, \ldots, u_l\rangle \mid (a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b'_j \ldots, a'_n, b'_n)$ is S-valid, and $\Theta(SEMI\_HULL^{n}_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b'_j\ldots,a'_n,b'_n}) = L^{Q'}_{u_1,u_2,\ldots,u_l}\}$.

Let $v_1 = \min(\{u_1 \mid (\exists u_2, \ldots, u_l)\langle u_1, u_2, \ldots, u_l\rangle \in X\})$.

For $1 < i < k$, let $v_i = \min(\{u_i \mid (\exists u_{i+1}, \ldots, u_l)\langle v_1, \ldots, v_{i-1}, u_i, \ldots, u_l\rangle \in X\})$.

Let $b'_j, \ldots, a'_n, b'_n$ be such that $\Theta(SEMI\_HULL^{n}_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b'_j\ldots,a'_n,b'_n}) = L^{Q'}_{v_1,\ldots,v_{k-1},u_k,\ldots,u_l}$ for some $u_k, \ldots, u_l$.

Let us fix a $b_j > \sum_{k=j}^{k=n} b'_k$, $b_j \in S$. Then, for any a''_w, b''_w, $j < w \leq n$, such that $(a_1, b_1, \ldots, a_j, b_j, a''_{j+1}, b''_{j+1}, \ldots, a''_n, b''_n)$ is S-valid, we have that

$\Theta(SEMI\_HULL^{n}_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j,a''_{j+1},b''_{j+1},\ldots,a''_n,b''_n}) = L^{Q'}_{v_1,\ldots,v_{k-1},u_k,\ldots,u_l}$, for some values of $u_k, \ldots, u_l$ (due to monotonicity of Θ; Note that $SEMI\_HULL^{j}_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j} \subseteq SEMI\_HULL^{n}_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b'_j\ldots,a'_n,b'_n}$ by Proposition 24, and, therefore, $SEMI\_HULL^{n}_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j,a''_{j+1},b''_{j+1},\ldots,a''_n,b''_n} \subseteq SEMI\_HULL^{n}_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b'_j\ldots,a'_n,b'_n}$, but $v_1, \ldots, v_{k-1}$ are chosen to be the minimum possible values, thus only $u_k, \ldots, u_l$ can vary for $a''_{j+1}, b''_{j+1}, \ldots, a''_n, b''_n$).

Thus, Θ (along with Ψ) essentially witnesses that

$\{SEMI\_HULL^{n}_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j,a''_{j+1},b''_{j+1},\ldots,a''_n,b''_n} \mid (a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j, a''_{j+1}, b''_{j+1}, \ldots, a''_n, b''_n)$ is S-valid $\}$ $\leq^{\mathbf{TxtEx}} \mathcal{L}^{Q'''}$ (since parameters for the first $k-1$ *INIT*s are fixed in Q' to be $v_1, \ldots, v_{k-1}$). This is not possible by induction hypothesis (since $\mathcal{L}^{Q''} \not\leq^{\mathbf{TxtEx}} \mathcal{L}^{Q'''}$). ∎

Corollary 6 *Suppose S is $\mathbf{rat}^+$-covering. Let $Q = (q_1, \ldots, q_{2n})$, where $q_{2i+1} = INIT$ and $q_{2i+2} = COINIT$, for $i < n$. Suppose $Q' \in BASIC^*$ is such that $\mathcal{L}^{Q} \not\leq \mathcal{L}^{Q'}$. Then, $SEMI\_HULL^{n,S} \not\leq^{\mathbf{TxtEx}} \mathcal{L}^{Q'}$.*

8 Q-classes Which Are Reducible to $SEMI\_HULL^{n,S}$

In this section we establish the best possible lower bound on the complexity of $SEMI\_HULL^{n,S}$ in terms of Q-classes.

We begin with two useful technical propositions.

Proposition 28 *Suppose S is* $\mathbf{rat}^+$*-covering. Then there exists a $S' \subseteq S$, such that S' is* $\mathbf{rat}^+$*-covering, and for all $b, b' \in S'$, if $b < b'$ then $2b < b'$.*

PROOF. Suppose h is an isomorphism from Z to S such that $h(i) < h(i+1)$. Define h' as follows: $h'(0) = h(0)$. Suppose we have defined $h'(i)$ and $h'(-i)$. Then define $h'(i+1)$ and $h'(-i-1)$ as follows. $h'(i+1) = h(j)$, such that $h(j) > 2h'(i)$, and $h'(-i-1) = h(j')$ such that $h(j') < h'(-i)/2$. Note that such j and j' exist since S is $\mathbf{rat}^+$*-covering*. It immediately follows that $S' = \text{range}(h')$ satisfies the requirements of the Proposition. ∎

Proposition 29 *Suppose $S' \subseteq S$. Then $SEMI\_HULL^{n,S'} \leq^{\mathbf{TxtEx}} SEMI\_HULL^{n,S}$.*

PROOF. Follows trivially, since $SEMI\_HULL^{n,S'} \subseteq SEMI\_HULL^{n,S}$. ∎

Theorem 5 *Suppose S is* $\mathbf{rat}^+$*-covering. Let $n \in N^+$, and $Q = (q_1, q_2, \ldots, q_n)$, where $q_1 = HALF$, and for $2 \leq i \leq n$, $q_i = INIT$.*

Then, $\mathcal{L}^Q \leq^{\mathbf{TxtEx}} SEMI\_HULL^{n,S}$.

PROOF. Intuitive idea of the reduction is as follows. Fix the first "break" point to be $a_1 = 1$. The first $HALF$-component can be reduced to the first slope b_1. Then the $(j-1)$-th *INIT*-component i_j is reduced to a_j. Then b_j is fixed based on a_j in such a way that $i_j < i'_j$ would imply $a_j < a'_j$ and $b_j > b'_j + \sum_{j<i\leq n} b'_i$, for any potential values of b'_i, $i > j$. This ensures the desired monotonicity in the definition of the reducing operator Θ.

Now we proceed with the formal proof.

Without loss of generality (using Propositions 28 and 29) we can assume that, for each $b, b' \in S$, if $b < b'$, then $2b < b'$. Let h be an isomorphism from Z to S such that $h(i) < h(i+1)$.

For $i_1 \in Z$, and $i_2, \ldots, i_n \in N$, let $map(i_1, i_2, \ldots, i_n) = (a_1, b_1, \ldots, a_n, b_n)$, where a_j, b_j, $1 \leq j \leq n$ are defined as follows. $a_1 = 1$, $b_1 = h(i_1 - a_1)$. Suppose we have defined $a_1, b_1, \ldots, a_k, b_k$. Then let $A_{k+1} = \{x \in N \mid x > a_k, [\sum_{1\leq i\leq k} b_i * (x \dot{-} a_i)] \in N\}$, and then let a_{k+1} to be the $(i_{k+1}+1)$-th least element in A_{k+1}. Let $b_{k+1} = h(i_1 - a_{k+1})$.

Claim 3 $(i_1, i_2, \ldots, i_n) <_Q (i'_1, i'_2, \ldots, i'_n)$ *implies* $SEMI\_HULL^{n,S}_{map(i_1,\ldots,i_n)} \subset SEMI\_HULL^{n,S}_{map(i'_1,\ldots,i'_n)}$.

PROOF. Suppose $(i_1, i_2, \ldots, i_n) <_Q (i'_1, i'_2, \ldots, i'_n)$. Suppose $map(i_1, \ldots, i_n) = (a_1, b_1, \ldots, a_n, b_n)$ and $map(i'_1, \ldots, i'_n) = (a'_1, b'_1, \ldots, a'_n, b'_n)$.

We consider the following cases.

Case 1: $i_1 > i'_1$.

In this case

(1) $b_1 > b'_1$, and

(2) $a_1 = a'_1 = 1$.

From (1) it follows that

(3) $b_1 > 2 * b'_1$.

Now, for $1 \leq i < n$, since $a'_i < a'_{i+1}$, and $b'_i = h(i'_1 - a'_i)$, we have $b'_i > b'_{i+1}$. Thus, $b'_i > 2 * b'_{i+1}$, for $1 \leq i < n$ by hypothesis about elements of S.

Thus, $\sum_{1\leq i\leq n} b'_i \leq 2b'_1$. Along with (3), we have that $b_1 \geq \sum_{1\leq i\leq n} b'_i$. This along with (2), Proposition 18 and Proposition 24 gives $SEMI\_HULL^{n,S}_{a_1,b_1,\ldots,a_n,b_n} \subseteq SEMI\_HULL^{1,S}_{a_1,b_1} \subseteq SEMI\_HULL^{n,S}_{a'_1,b'_1,\ldots,a'_n,b'_n}$.

Case 2: For some j, $1 < j \leq n$, for $1 \leq k < j$, $i_k = i'_k$, but $i_j < i'_j$.

In this case

(4) $a_i = a'_i$ and $b_i = b'_i$, for $1 \leq i < j$.

(5) $b_j > b'_j$, and

(6) $a_j < a'_j$.

From (5) it follows that

(7) $b_j > 2 * b'_j$.

Now, for $1 \leq i < n$, since $a'_i < a'_{i+1}$, and $b'_i = h(i'_1 - a'_i)$, we have $b'_i > b'_{i+1}$. Thus, $b'_i > 2 * b'_{i+1}$, for $1 \leq i < n$ by hypothesis about elements of S.

Thus, $\sum_{j \leq i \leq n} b'_i \leq 2b'_j$. This, along with (7) gives us that $b_j > \sum_{j \leq i \leq n} b'_i$. Thus using (6), Proposition 18 and Proposition 24 we have $SEMI\_HULL^{n,S}_{a_1,b_1,\ldots,a_n,b_n} \subseteq SEMI\_HULL^{j,S}_{a_1,b_1,\ldots,a_j,b_j} \subseteq SEMI\_HULL^{n,S}_{a'_1,b'_1,\ldots,a'_n,b'_n}$.

Claim follows from above cases. $\square$

For any $X \subseteq N$, let $\Theta(X) = \bigcup_{\langle i_1,\ldots,i_n\rangle \in X} SEMI\_HULL^n_{map(i_1,\ldots,i_n)}$.

Thus, it follows that $\Theta(L^Q_{i_1,i_2,\ldots,i_n}) = SEMI\_HULL^{n,S}_{map(i_1,\ldots,i_n)}$.

Define Ψ as follows. If a sequence α of grammar converges to a grammar for $SEMI\_HULL^{n,S}_{map(i_1,\ldots,i_n)}$, then $\Psi(\alpha)$ converges to a grammar for $L^Q_{i_1,i_2,\ldots,i_n}$.

It is now easy to verify that Θ and Ψ witness that $\mathcal{L}^Q \leq^{\mathbf{TxtEx}} SEMI\_HULL^{n,S}$. ∎

The following technical result will be used to show that Q-classes with $n+1$ *INIT*s in Q cannot be reduced to $SEMI\_HULL^{n,S}$.

Theorem 6 *Suppose* $0 \leq j \leq n$. $Q = (q_1, q_2, \ldots, q_{n+1-j})$, *where* $q_i = INIT$, *for* $1 \leq i \leq n-j+1$. *Let* $R = R_1 \times R_2 \times \ldots \times R_{n+1-j}$, *where* R_1 *is of cardinality at least* 2 *and, for* $2 \leq i \leq n+2-j$, R_i *is an infinite subset of* N. *Suppose* S *is* $\mathbf{rat}^+$*-covering, and* $(a_1, b_1, \ldots, a_j, b_j)$ *is* S*-valid.*

Then,

$\mathcal{L}^{Q,R} \not\leq^{\mathbf{TxtEx}} \{SEMI\_HULL^n_{a_1,b_1,\ldots,a_j,b_j,a'_{j+1},b'_{j+1},\ldots,a'_n,b'_n} \mid (a_1, b_1, \ldots, a_j, b_j, a'_{j+1}, b'_{j+1}, \ldots, a'_n, b'_n)$ *is* S*-valid* $\}$.

Proof. We prove the theorem by reverse induction on j (from n to 0). For $j = n$, theorem clearly holds (since $\mathcal{L}^{Q,R}$ contains at least two languages, whereas $\{SEMI\_HULL^n_{a_1,b_1,\ldots,a_j,b_j,a'_{j+1},b'_{j+1},\ldots,a'_n,b'_n} \mid (a_1, b_1, \ldots, a_j, b_j, a'_{j+1}, b'_{j+1}, \ldots, a'_n, b'_n)$ is S-valid $\}$ contains only one language).

So suppose the theorem holds for $j' < j \leq n$. Then, we show that the theorem holds for $j = j'$.

Suppose by way of contradiction that Θ (along with some Ψ) witnesses that $\mathcal{L}^{Q,R} \leq^{\mathbf{TxtEx}} \{SEMI\_HULL^n_{a_1,b_1,\ldots,a_j,b_j,a'_{j+1},b'_{j+1},\ldots,a'_n,b'_n} \mid (a_1, b_1, \ldots, a_j, b_j, a'_{j+1}, b'_{j+1}, \ldots, a'_n, b'_n)$ is S-valid $\}$, for some S-valid $(a_1, b_1, \ldots, a_j, b_j)$.

Let w be the minimal element in R_1. For $i \in R_2$, suppose $\Theta(L^Q_{w,i,\min(R_3),\min(R_4),\ldots}) = SEMI\_HULL^{n,b}_{a_1,b_1,\ldots,a_j,b_j,a_{j+1}(i),b_{j+1}(i),\ldots,a_n(i),b_n(i)}$. If, as i varies, $a_{j+1}(i)$ takes arbitrary large value then, for $w' > w$, $w' \in R_1$, $\Theta(L^Q_{w',\min(R_2),\min(R_3),\ldots}) = SEMI\_HULL^j_{a_1,b_1,\ldots,a_j,b_j}$ and is thus not a member of $\{SEMI\_HULL^n_{a_1,b_1,\ldots,a_j,b_j,a'_{j+1},b'_{j+1},\ldots,a'_n,b'_n} \mid (a_1, b_1, \ldots, a_j, b_j, a'_{j+1}, b'_{j+1}, \ldots, a'_n, b'_n)$ is S-valid $\}$.

So suppose $i_1 \in R_2$ maximizes $a_{j+1}(i_1)$. Now for $i > i_1$, if $b_{j+1}(i)$ takes arbitrary small value, then for $w' > w$, $\Theta(L^Q_{w',\min(R_2),\min(R_3),\ldots}) \supseteq \bigcup_{b'_{j+1} \in S} SEMI\_HULL^j_{a_1,b_1,\ldots,a_j,b_j,a_{j+1}(i_1),b'_{j+1}}$. However, any member of $\{SEMI\_HULL^n_{a_1,b_1,\ldots,a_j,b_j,a'_{j+1},b'_{j+1},\ldots,a'_n,b'_n} \mid (a_1,b_1,\ldots,a_j,b_j,a'_{j+1},b'_{j+1},\ldots,a'_n,b'_n)$ is S-valid$\}$, misses out infinitely many elements in $\bigcup_{b'_{j+1}\in S} SEMI\_HULL^j_{a_1,b_1,\ldots,a_j,b_j,a_{j+1}(i_1),b'_{j+1}}$. Thus, $\Theta(L^Q_{w',\min(R_2),\min(R_3),\ldots}) \notin \{SEMI\_HULL^n_{a_1,b_1,\ldots,a_j,b_j,a'_{j+1},b'_{j+1},\ldots,a'_n,b'_n} \mid (a_1,b_1,\ldots,a_j,b_j,a'_{j+1},b'_{j+1},\ldots,a'_n,b'_n)$ is S-valid$\}$.

So, suppose $i_2 > i_1$, $i_2 \in R_2$, minimizes $b_1(i_2)$.

Thus, Θ (along with Ψ) essentially witnesses that that $\mathcal{L}^{Q',R'} \leq^{\mathbf{TxtEx}} \{SEMI\_HULL^n_{a_1,b_1,\ldots,a_j,b_j,a_{j+1}(i_2),b_{j+1}(i_2),a'_{j+2},b'_{j+2},\ldots,a'_n,b'_n} \mid (a_1,b_1,\ldots,a_j,b_j,a_{j+1}(i_2),b_{j+1}(i_2),a'_{j+2},b'_{j+2},\ldots,a'_n,b'_n)$ is S-valid$\}$, where Q' is obtained by dropping q_1 from Q. R' is obtained from R by dropping R_1 and changing R_2 to $R_2-\{x \mid x \leq i_2\}$.

This is a contradiction to induction hypothesis. ∎

Corollary 7 *Suppose $Q = (q_1, q_2, \ldots, q_{n+1})$, where $q_i = INIT$, for $1 \leq i \leq n+1$. Suppose S is* **rat**$^+$*-covering.*

Then, $\mathcal{L}^Q \not\leq^{\mathbf{TxtEx}} SEMI\_HULL^{n,S}$.

Proof similar to one used in proving Theorem 6 above can be used to show the following theorem (we just need to interchange the role of w and w' in the proof, for $j=0$ case).

Theorem 7 *Let $n \in N^+$, and $Q = (q_1, q_2, \ldots, q_{n+1})$, where $q_1 = COINIT$, and $q_i = INIT$, for $2 \leq i \leq n+1$. Let $R = R_1 \times R_2 \times \ldots \times R_{n+1}$, where R_1 is of cardinality at least 2 and, for $2 \leq i \leq n+2-j$, R_i is an infinite subset of N. Suppose S is any* **rat**$^+$*-covering set.*

Then, $\mathcal{L}^{Q,R} \not\leq^{\mathbf{TxtEx}} SEMI\_HULL^{n,S}$.

Corollary 8 *Suppose $Q = (q_1, q_2, \ldots, q_{n+1})$, where $q_1 = COINIT$, and $q_i = INIT$, for $2 \leq i \leq n+1$. Suppose S is* **rat**$^+$*-covering.*

Then, $\mathcal{L}^Q \not\leq^{\mathbf{TxtEx}} SEMI\_HULL^{n,S}$.

Do there exist Q-classes reducible to $SEMI\_HULL^{n,S}$ with $COINIT$ on the second or greater positions in Q? Our next results show that even $(COINIT, COINIT)$ and $(INIT, COINIT)$ are not reducible to $SEMI\_HULL^{n,S}$.

Based on Corollary 3, let us define $<_{valid}$ as follows.

Definition 19 Suppose $(a_1, b_1, \ldots, a_n, b_n)$ and $(a'_1, b'_1, \ldots, a'_n, b'_n)$ are valid. Then, $(a_1, b_1, \ldots, a_n, b_n) <_{valid} (a'_1, b'_1, \ldots, a'_n, b'_n)$ iff there exists an i, $1 \leq i \leq n$ such that, for $1 \leq j < i$, $a_j = a'_j$, $b_j = b'_j$ and $a_i < a'_i$ or $a_i = a'_i$ and $b_i > b'_i$.

Note that $<_{valid}$ imposes a total order among valid sequences of form $(a_1, b_1, \ldots, a_n, b_n)$. Moreover, $SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} \subset SEMI\_HULL^n_{a'_1,b'_1,\ldots,a'_n,b'_n}$, implies $(a_1, b_1, \ldots, a_n, b_n) <_{valid} (a'_1, b'_1, \ldots, a'_n, b'_n)$.

Similarly define $(a_1, b_1, \ldots, a_n, b_n) \leq_{valid} (a'_1, b'_1, \ldots, a'_n, b'_n)$ iff $(a_1, b_1, \ldots, a_n, b_n) <_{valid} (a'_1, b'_1, \ldots, a'_n, b'_n)$ or $(a_1, b_1, \ldots, a_n, b_n) = (a'_1, b'_1, \ldots, a'_n, b'_n)$.

$\geq_{valid}$ and $>_{valid}$ can be similarly defined.

For any fixed $n \in N$, $X \subseteq \{(a_1, b_1, \ldots, a_n, b_n) \mid (a_1, b_1, \ldots, a_n, b_n) \in VALID_S\}$, let $\max_{valid}(X) = (a_1, b_1, \ldots, a_n, b_n) \in X$, if any, such that $(\forall (a'_1, b'_1, \ldots, a'_n, b'_n) \in X)[(a'_1, b'_1, \ldots, a'_n, b'_n) \leq_{valid} (a_1, b_1, \ldots, a_n, b_n)]$.

Similarly, let $\min_{valid}(X) = (a_1, b_1, \ldots, a_n, b_n) \in X$, if any, such that $(\forall (a'_1, b'_1, \ldots, a'_n, b'_n) \in X)[(a_1, b_1, \ldots, a_n, b_n) \leq_{valid} (a'_1, b'_1, \ldots, a'_n, b'_n)]$.

Proposition 30 *Suppose $n \in N$, S is* **rat**$^+$*-covering, and $X \subseteq \{(a_1, b_1, \ldots, a_n, b_n) \mid (a_1, b_1, \ldots, a_n, b_n) \in VALID_S\}$. Then, if $\min_{valid}(X)$ does not exist, then no subset of $(\bigcap_{I\in X} SEMI\_HULL^n_I)$ belongs to $SEMI\_HULL^{n,S}$.*

PROOF. Suppose S, X is given as above, and $\min_{valid}(X)$ does not exist. Then, there exists a j, such that for some $a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j$, there exist arbitrary large b_j such that $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j, \ldots) \in X$, for some $a_{j+1}, b_{j+1}, \ldots, a_n, b_n$, (which may depend on b_j). Thus, $\bigcap_{I \in X} SEMI\_HULL_I^n \subseteq \{(x, y) \mid x \leq a_j\}$. But no subset of $\{(x, y) \in N^2 \mid x \leq a_j\}$ belongs to $SEMI\_HULL^{n,S}$ (any language L in $SEMI\_HULL^{n,S}$ satisfies: for all x, there exists a y such that $(x, y) \in L$). ∎

Theorem 8 *Suppose S is* **rat**$^+$*-covering and $n \in N$. Let $Q = (COINIT, COINIT)$, $R = R_1 \times R_2$, where R_2 is infinite, and R_1 contains at least two elements. Then $\mathcal{L}^{Q,R} \not\leq^{\mathbf{TxtEx}} SEMI\_HULL^{n,S}$.*

PROOF. Suppose by way of contradiction that $\mathcal{L}^{Q,R} \leq^{\mathbf{TxtEx}} SEMI\_HULL^{n,S}$ as witnessed by Θ (along with Ψ).

Let $w = \min(R_1)$. Suppose $\Theta(L_{w,x}^Q) = SEMI\_HULL^n_{a_1(x), b_1(x), \ldots a_n(x), b_n(x)}$, for $x \in R_2$.

Note that, for $x < x'$, $x, x' \in R_2$, $\Theta(L_{w,x}^Q) \supset \Theta(L_{w,x'}^Q)$. Thus, for $x < x'$, $x, x' \in R_2$, $(a_1(x), \ldots, b_n(x)) >_{valid} (a_1(x'), \ldots, b_n(x'))$, by Corollary 3.

Let $X = \{(a_1(x), b_1(x), \ldots, a_n(x), b_n(x)) \mid x \in R_2\}$.

If $\min_{valid}(X)$ does not exist, then for any $w_1 > w$, $w_1 \in R_1$ and $w_2 \in R_2$, $\Theta(L_{w_1,w_2}^Q) \subseteq \bigcap_{x \in R_2} SEMI\_HULL^n_{a_1(x), b_1(x), \ldots}$, does not belong to $SEMI\_HULL^{n,S}$ by Proposition 30.

So suppose $x_1 \in R_2$ is such that $\min_{valid}(X) = (a_1(x_1), b_1(x_1), \ldots, a_n(x_1), b_n(x_1))$. But then, for all $x \geq x_1$, $(a_1(x), \ldots, b_n(x)) \leq_{valid} (a_1(x_1), \ldots, b_n(x_1))$ (due to monotonicity of Θ). Thus, for all $x \geq x_1$, $x \in R_2$, $(a_1(x), \ldots, b_n(x)) = (a_1(x_1), \ldots, b_n(x_1))$. A contradiction to Θ (along with Ψ) witnessing that $\mathcal{L}^{Q,R} \leq^{\mathbf{TxtEx}} SEMI\_HULL^{n,S}$. ∎

Corollary 9 *Suppose S is* **rat**$^+$*-covering, and $n \in N$. Let $Q = (COINIT, COINIT)$. Then $\mathcal{L}^Q \not\leq^{\mathbf{TxtEx}} SEMI\_HULL^{n,S}$.*

Similarly, to the proof of Theorem 8 we also have the following theorem (we just need to interchange the roles of $L_{w,x}^Q$ and $L_{w',x}^Q$ in above proof).

Theorem 9 *Suppose S is* **rat**$^+$*-covering. Let $Q = (INIT, COINIT)$, $R = R_1 \times R_2$, where R_2 is infinite, and R_1 contains at least two elements. Then $\mathcal{L}^{Q,R} \not\leq^{\mathbf{TxtEx}} SEMI\_HULL^{n,S}$.*

Corollary 10 *Suppose S is* **rat**$^+$*-covering. Let $Q = (INIT, COINIT)$. Then $\mathcal{L}^Q \not\leq^{\mathbf{TxtEx}} SEMI\_HULL^{n,S}$.*

9 Definitions for Complements of Open Semi Hull

In this section we define the classes of complements of *SEMI_HULL*s and establish some useful propositions following from appropriate definitions.

Definition 20 Suppose $a_1, \ldots, a_n \in N$ and $b_1, \ldots, b_n \in$ **rat**$^+$, where $0 < a_1 < a_2 < \ldots < a_n$.

$coSEMI\_HULL^n_{a_1,b_1,a_2,b_2,\ldots,a_n,b_n} = \{(x, y) \in N^2 \mid y < \sum_{1 \leq i \leq n} b_i * (x \dot{-} a_i)\} = N^2 - SEMI\_HULL^n_{a_1,b_1,a_2,b_2,\ldots,a_n,b_n}$.

Definition 21 Suppose $S \subseteq$ **rat**$^+$ is **rat**$^+$*-covering*. $coSEMI\_HULL^{n,S} = \{coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} \mid (a_1, b_1, \ldots, a_n, b_n) \in VALID_S\}$.

Proposition 31 *Suppose* $(a_1, b_1, \ldots, a_n, b_n)$ *and* $(a'_1, b'_1, \ldots, a'_m, b'_m)$ *are valid.*

Then, $\mathit{SEMI\_HULL}^n_{a_1,b_1,\ldots,a_n,b_n} \subseteq \mathit{SEMI\_HULL}^m_{a'_1,b'_1,\ldots,a'_m,b'_m}$ *iff* $\mathit{coSEMI\_HULL}^n_{a_1,b_1,\ldots,a_n,b_n} \supseteq \mathit{coSEMI\_HULL}^m_{a'_1,b'_1,\ldots,a'_m,b'_m}$

PROOF. Follows from the definitions. ∎

Definition 22 Suppose $a_1, b_1, \ldots,, a_j, b_j$ are given such that $(a_1, b_1, \ldots, a_j, b_j) \in \mathit{VALID}$. Then, let $\mathit{coINTER}(a_1, b_1, \ldots, a_j, b_j) = \bigcup\{\mathit{coSEMI\_HULL}^n_{a_1,b_1,\ldots,a_j,b_j,\ldots,a_n,b_n} \mid n \geq j \wedge (a_1, b_1, \ldots, a_j, b_j, \ldots, a_n, b_n) \in \mathit{VALID}\} = N^2 - \mathit{INTER}(a_1, b_1, \ldots, a_j, b_j)$.

Proposition 32 *Suppose* $1 \leq j \leq n$, *and* $(a_1, b_1, \ldots, \ldots, a_n, b_n)$ *and* $(a'_1, b'_1, \ldots, a'_m, b'_m)$ *are valid. Then* $\mathit{coINTER}(a'_1, b'_1, \ldots, a'_m, b'_m) \supseteq \mathit{coSEMI\_HULL}^n_{a_1,b_1,\ldots,a_n,b_n}$ *iff* $\mathit{INTER}(a'_1, b'_1, \ldots, a'_m, b'_m) \subseteq \mathit{SEMI\_HULL}^n_{a_1,b_1,\ldots,a_n,b_n}$.

PROOF. Follows from the definitions. ∎

Proposition 33 *Suppose* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1})$ *is valid. Let* $(x, y) \in N^2$ *be such that* $y > \sum_{1 \leq i < j} b_i(x \dot{-} a_i)$. *Then, there exists a* $B' \in \mathbf{rat}^+$ *obtainable effectively from* $a_1, b_1, \ldots, a_{j-1}, b_{j-1}, x$ *and* y *such that for any* (a'_j, b'_j), *if* (a'_j, b'_j), *is valid and* $(x, y) \in \mathit{coINTER}^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j}$, *then* $a'_j < x$ *and* $b'_j \geq B'$.

PROOF. Follows from Proposition 25. ∎

10 Classes to Which $\mathit{coSEMI\_HULL}^{n,S}$ is Reducible

In this section we obtain nearly the best possible upper bound on complexity of *coSEMI_HULL*s in terms of Q-degrees. Intuitive upper bound $\mathcal{L}^Q$ for $Q = (q_1, q_2, \ldots, q_{2n-1}, q_{2n})$ with $q_{2i+1} = \mathit{COINIT}$ and $q_{2i+2} = \mathit{HALF}$ can be easily established using the following learning strategy for $\mathit{coSEMI\_HULL}^{n,S}$: apply a *COINIT*-type strategy to learn the first "break" point a_1, then apply a *HALF*-type strategy to learn the first slope b_1, etc. However, the optimal upper bound established below contains only $n + 1$ components!

Theorem 10 *Suppose* S *is* $\mathbf{rat}^+$*-covering. Suppose* $Q = (q_1, q_2, \ldots, q_{n+1})$, *where* $q_1 = \mathit{INIT}$ *and* $q_i = \mathit{COINIT}$, *for* $2 \leq i \leq n + 1$. *Then* $\mathit{coSEMI\_HULL}^{n,S} \leq^{\mathbf{TxtEx}} \mathcal{L}^Q$.

PROOF. The intuitive strategy for learning $\mathit{coSEMI\_HULL}^{n,S}$ providing the desired upper bound operates as follows: first, it applies an *INIT*-type strategy to learn a bound on the maximum slope: $\sum_i b_i$; then, using this bound, it applies *COINIT*-type strategies to learn every pair of parameters a_i, b_i. This *COINIT*-type strategy is in some sense mirror image of the *INIT*-type strategy used in the proof of Theorem 3. The technical details though become somewhat more complicated.

Now we proceed with the formal proof.

Fix S which is $\mathbf{rat}^+$*-covering*. Let h be a recursive bijection from Z to S such that $h(i) < h(i + 1)$. Fix code as in Proposition 27.

Claim 4 *Suppose* $B \in S$. *Suppose* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j), (a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \in \mathit{VALID}_S$, *where* $b_i, b'_i \leq B$, *and* $(a_j, b_j) \neq (a'_j, b'_j)$. *Suppose* $\mathit{coINTER}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \supseteq \mathit{coSEMI\_HULL}^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$.

Then, $\mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) < \mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$.

PROOF. By Proposition 23 and definitions of *SEMI_HULL*, *coSEMI_HULL*, *INTER*, *coINTER*, we have that $a'_j \leq a_j$.

Now, since, $coINTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \supseteq coSEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$, we have $INTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq SEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$. Thus, we have $\mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) < \mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$ (by Proposition 27; for getting $<$ instead of $\leq$ use the fact that $(a_j, b_j) \neq (a'_j, b'_j)$). $\square$

Let *map* be a mapping from *VALID* to N^* such that $map(a_1, b_1, \ldots, a_n, b_n) = (h^{-1}(B), \mathrm{code}(B, a_1, b_1), \mathrm{code}(B, a_1, b_1, a_2, b_2), \ldots, \mathrm{code}(B, a_1, b_1, \ldots, a_n, b_n))$, where B is the least element of S such that $\max(h(0), \sum_{1\leq i\leq n} b_i) \leq B$.

Claim 5 *Suppose* $(a_1, b_1, \ldots, a_n, b_n)$ *is* S*-valid. Let* $B = \min(\{b \in S \mid b \geq \max(h(0), \sum_{1\leq i\leq n} b_i)\})$.

(A) Suppose $1 \leq j \leq n$. *Suppose* $B \geq b'_j$, *and* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ *is* S*-valid. If* $\mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) > \mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$. *Then* $coINTER(a_1, b_1, \ldots, a'_j, b'_j) \not\supseteq coSEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$.

(B) Suppose $(a'_1, b'_1, \ldots, a'_n, b'_n)$ *is* S*-valid, and* $map(a'_1, b'_1, \ldots, a'_n, b'_n) <_Q map(a_1, b_1, \ldots, a_n, b_n)$. *Then either* $B > \min(\{b \in S \mid b \geq \max(h(0), \sum_{1\leq i\leq n} b'_i)\})$ *or for the least* j, $1 \leq j \leq n$, *such that* $(a_j, b_j) \neq (a'_j, b'_j)$, $coINTER(a'_1, b'_1, \ldots, a'_j, b'_j) \not\supseteq coSEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$.

(C) Suppose $(a'_1, b'_1, \ldots, a'_n, b'_n)$ *is* S*-valid, and* $map(a'_1, b'_1, \ldots, a'_n, b'_n) <_Q map(a_1, b_1, \ldots, a_n, b_n)$. *Then either* $B > \min(\{b \in S \mid b \geq \max(h(0), \sum_{1\leq i\leq n} b'_i)\})$ *or* $coINTER(a'_1, b'_1, \ldots, a'_j, b'_j, \ldots, a'_n, b'_n) \not\supseteq coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}$.

(D) If $coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} \subset coSEMI\_HULL^n_{a'_1,b'_1,\ldots,a'_n,b'_n}$, *then* $map(a_1, b_1, \ldots, a_n, b_n) <_Q map(a'_1, b'_1, \ldots, a'_n, b'_n)$.

(E) Suppose $1 \leq j \leq n$. *There exists a finite* $X_j \subseteq coSEMI\_HULL^n_{a_1,b_1,\ldots,a_j,b_j}$ *such that, for all* S*-valid* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a''_j, b''_j)$, *if* $B \geq b''_j$ *and* $\mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a''_j, b''_j) > \mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$, *then* $coINTER(a_1, b_1, \ldots, a''_j, b''_j) \not\supseteq X_j$.

(F) There exists a finite $X \subseteq coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}$ *such that, for all* S*-valid* $(a'_1, b'_1, \ldots, a'_n, b'_n)$, *if* $B = \min(\{b \in S \mid b \geq \max(h(0), \sum_{1\leq i\leq n} b'_i)\})$, *and* $map(a'_1, b'_1, \ldots, a'_n, b'_n) <_Q map(a_1, b_1, \ldots, a_n, b_n)$, *then* $coINTER(a'_1, b'_1, \ldots, a'_n, b'_n) \not\supseteq X$.

PROOF. (A) Follows from Claim 4.

(B) Let $B' = \min(\{b \in S \mid b \geq \max(h(0), \sum_{1\leq i\leq n} b'_i)\})$. Since $map(a'_1, b'_1, \ldots, a'_n, b'_n) <_Q map(a_1, b_1, \ldots, a_n, b_n)$, $h^{-1}(B') \leq h^{-1}(B)$. Therefore, $B' \leq B$, and we must have $\min(\{b \in S \mid b \geq \max(h(0), \sum_{1\leq i\leq n} b'_i)\}) \leq \min(\{b \in S \mid b \geq \max(h(0), \sum_{1\leq i\leq n} b_i)\}) = B$. If $\min(\{b \in S \mid b \geq \max(h(0), \sum_{1\leq i\leq n} b'_i)\}) = B$, then let j be the least value such that $1 \leq j \leq n$, and $(a'_j, b'_j) \neq (a_j, b_j)$. Thus, $\mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) > \mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$. Now part (B) follows from part (A).

(C) Since $coINTER(a'_1, b'_1, \ldots, a'_j, b'_j) \supseteq coINTER(a'_1, b'_1, \ldots, a'_j, b'_j, \ldots, a'_n, b'_n)$ (by Proposition 21 and definitions of *INTER* and *coINTER*), and $coSEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j} \subseteq coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}$ (Proposition 18 and definitions of *SEMI_HULL*, *coSEMI_HULL*), part (C) follows from part (B).

(D) Suppose $coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} \subset coSEMI\_HULL^n_{a'_1,b'_1,\ldots,a'_n,b'_n}$. Thus, we must have $\min(\{b \in S \mid b \geq \max(h(0), \sum_{1\leq i\leq n} b'_i)\}) \geq \min(\{b \in S \mid b \geq \max(h(0), \sum_{1\leq i\leq n} b_i)\})$ (since otherwise, for all but finitely many x, $\sum_{1\leq i\leq n} b_i(x \dot{-} a_i) > 1 + \sum_{1\leq i\leq n} b'_i(x \dot{-} a'_i)$, which would contradict $coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} \subseteq coSEMI\_HULL^n_{a'_1,b'_1,\ldots,a'_n,b'_n}$).

If $\min(\{b \in S \mid b \geq \max(h(0), \sum_{1\leq i\leq n} b'_i)\}) > \min(\{b \in S \mid b \geq \max(h(0), \sum_{1\leq i\leq n} b_i)\})$, then clearly, $map(a_1, b_1, \ldots, a_n, b_n) <_Q map(a'_1, b'_1, \ldots, a'_n, b'_n)$. If $\min(\{b \in S \mid b \geq \max(h(0), \sum_{1\leq i\leq n} b'_i)\}) = \min(\{b \in S \mid b \geq \max(h(0), \sum_{1\leq i\leq n} b_i)\})$, then (D) follows from part (C), and the fact that $coSEMI\_HULL^n_{a'_1,b'_1,\ldots,a'_n,b'_n} \subseteq coINTER(a'_1, b'_1, \ldots, a'_n, b'_n)$.

(E) Let $(x, y) \in N^2$ be such that $(x, y) \in coSEMI\_HULL^n_{a_1,b_1,\ldots,a_j,b_j} \subseteq coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}$, but $y > \sum_{1\leq i<j} b_i(x \dot{-} a_i)$. Note that there exists such (x, y).

By

Proposition 33, there exists a $B' \in \mathbf{rat}^+$ such that, if $(x, y) \in coINTER^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a''_j,b''_j}$, then $a''_j < x$ and $b'_j \geq B'$. Thus, it follows that for any (a''_j, b''_j), if $a''_j \geq x$, or $b''_j < B'$, $(x, y) \notin coINTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a''_j, b''_j)$. Now, for each $a''_j < x$, and $b''_j \in S$ such that $B' \leq b''_j \leq B$, if $coSEMI\_HULL^j_{a_1,b_1,\ldots,a_j,b_j} \not\subseteq coINTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a''_j, b''_j)$, then pick $x_{a''_j,b''_j}, y_{a''_j,b''_j}$ such that $(x_{a''_j,b''_j}, y_{a''_j,b''_j}) \in coSEMI\_HULL^j_{a_1,b_1,\ldots,a_j,b_j} - coINTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a''_j, b''_j)$; if $coSEMI\_HULL^j_{a_1,b_1,\ldots,a_j,b_j} \subseteq coINTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a''_j, b''_j)$, then let $(x_{a''_j,b''_j}, y_{a''_j,b''_j}) = (x, y)$.

Now let $X_j = \{(x, y)\} \cup \{(x_{a''_j,b''_j}, y_{a''_j,b''_j}) \mid a''_j < x, B' \leq b''_j \leq B, b''_j \in S\}$. Now using part (A), it is now easy to verify that X_j witnesses the claim of part (E).

(F) Let $X = \bigcup_{1\leq j\leq n} X_j$ where X_j is as in part (E). Now, if $map(a'_1, b'_1, \ldots, a'_n, b'_n) <_Q map(a_1, b_1, \ldots, a_n, b_n)$, and $B = \min(\{b \in S \mid b \geq \max(h(0), \sum_{1\leq i\leq n} b'_i)\})$, then there exists a j, $1 \leq j \leq n$ such for $1 \leq i < j$, $a_i = a'_i$ and $b_i = b'_i$ and $\mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) > \mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$. Part (F) now follows from part (E) and definition of X. □

We now continue with the proof of the theorem. The aim is to construct Θ which maps $coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}$ to $L^Q_{map(a_1,b_1,\ldots,a_n,b_n)}$.

Note that definition of Ψ mapping grammar sequence converging to a grammar for $L^Q_{map(a_1,b_1,\ldots,a_n,b_n)}$ to a grammar sequence converging to a grammar for $coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}$ would be trivial. We thus just define Θ.

Without loss of generality, we will be giving Θ as mapping sets to sets.

For any finite $X \subseteq N^2$, let $Prop(X, a_1, b_1, \ldots, a_n, b_n)$ be true iff following three properties are satisfied. Let $B = \min(\{b \in S \mid b \geq h(0) \wedge (\forall (x, y) \in X)[b \geq \frac{y}{x}]\})$.

(A) $(a_1, b_1, \ldots, a_n, b_n) \in VALID_S$,

(B) $B \geq \min(\{b \in S \mid b \geq \max(h(0), \sum_{1\leq i\leq n} b_i)\})$

and

(C) If $B = \min(\{b \in S \mid b \geq \max(h(0), \sum_{1\leq i\leq n} b_i)\})$, then for all j, $1 \leq j \leq n$, for all $a'_j \in N, b'_j \in S$ such that $(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ is S-valid, $B \geq b'_j$ and $\mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) > \mathrm{code}(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j))$, $[X \not\subseteq coINTER_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j}]$.

Note that whether $X, a_1, b_1, \ldots, a_n, b_n$, satisfy (A) and (B) and (C) is effectively testable.

Moreover, (C) along with definition of B above implies that

(D) For all $(a'_1, b'_1, \ldots, a'_n, b'_n) \in VALID$ such that $b'_i \in S$ and $map(a'_1, b'_1, \ldots, a'_n, b'_n) <_Q map(a_1, b_1, \ldots, a_n, b_n)$, $X \not\subseteq coSEMI\_HULL(a'_1, b'_1, \ldots, a'_n, b'_n)$.

Thus for finite $X \subseteq N^2$, let

$\Theta(X) = \bigcup\{L^Q_{map(a_1,b_1,\ldots,a_n,b_n)} \mid Prop(X, a_1, b_1, \ldots, a_n, b_n)\}$.

For infinite X', $\Theta(X') = \bigcup_{X\subseteq X', \mathrm{card}(X)<\infty} \Theta(X)$.

It is easy to verify that

(1) for any $X \subseteq coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}$,

$\Theta(X) \subseteq L^Q_{map(a_1,b_1,\ldots,a_n,b_n)}$ (by (D) and the fact that for any valid I and I', $map(I) <_Q map(I')$, implies $L^Q_{map(I)} \subseteq L^Q_{map(I')}$), and

(2) For all S-valid $(a_1, b_1, \ldots, a_n, b_n)$, by Claim 5(F), there exists a finite $X \subseteq coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}$, such that $\Theta(X) \supseteq L^Q_{map(a_1,b_1,\ldots,a_n,b_n)}$.

Thus, we have that $\Theta(coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}) = L^Q_{map(a_1,b_1,\ldots,a_n,b_n)}$. ∎

The construction in the above proof can be slightly changed along the following lines: instead of learning first the bound on the maximum "slope", one can first apply a *COINIT*-type strategy trying to learn the parameters a_1, b_1 under assumption that b_1 is smaller than some fixed bound $B \in \mathbf{rat}^+$, and then apply an *INIT*-type strategy to learn both the bound on the maximum "slope" and b_1 if the latter becomes greater than the bound B. Thus we obtain the following

Theorem 11 *Suppose S is $\mathbf{rat}^+$-covering. Suppose $Q = (q_1, q_2, \ldots, q_{n+1})$, where $q_2 = INIT$ and $q_i = COINIT$, for $1 \leq i \leq n+1$, $i \neq 2$. Then $coSEMI\_HULL^{n,S} \leq^{\mathbf{TxtEx}} \mathcal{L}^Q$.*

11 Classes Which are Reducible to $coSEMI\_HULL^{n,S}$

In this section we will get a lower bound for $coSEMI\_HULL^{n,S}$ having n components, and thus being very close to the upper bounds obtained in the previous section.

Proposition 34 *Suppose $S' \subseteq S$. Then $coSEMI\_HULL^{n,S'} \leq^{\mathbf{TxtEx}} coSEMI\_HULL^{n,S}$.*

The proof of the following theorem is similar to the proof of the lower bound for *SEMI_HULL*s, with *COINIT*s replacing *INIT*s.

Theorem 12 *Suppose S is $\mathbf{rat}^+$-covering. Let $n \in N^+$, and $Q = (q_1, q_2, \ldots, q_n)$, where $q_1 = HALF$, and for $2 \leq i \leq n$, $q_i = COINIT$.*
Then, $\mathcal{L}^Q \leq^{\mathbf{TxtEx}} coSEMI\_HULL^{n,S}$.

Proof. Without loss of generality (using Propositions 28 and 34) we can assume that, for each $b, b' \in S$, if $b < b'$, then $2b < b'$. Let h be an isomorphism from Z to S such that $h(i) < h(i+1)$.

Let $map(i_1, i_2, \ldots, i_n) = (a_1, b_1, \ldots, a_n, b_n)$, where a_j, b_j, $1 \leq j \leq n$ are defined as follows. $a_1 = 1$, $b_1 = h(-i_1 - a_1)$. Suppose we have defined $a_1, b_1, \ldots, a_k, b_k$. Then let $A_{k+1} = \{x \in N \mid x > a_k, [\sum_{1\leq i\leq k} b_i * (x \dot{-} a_i)] \in N\}$, and then let a_{k+1} to be the $(i_{k+1}+1)$-th least element in A_{k+1}. Let $b_{k+1} = h(-i_1 - a_{k+1})$.

Claim 6 $(i_1, i_2, \ldots, i_n) <_Q (i'_1, i'_2, \ldots, i'_n)$ *implies* $coSEMI\_HULL^{n,S}_{map(i_1,\ldots,i_n)} \subset coSEMI\_HULL^{n,S}_{map(i'_1,\ldots,i'_n)}$.

Proof. Suppose $(i_1, i_2, \ldots, i_n) <_Q (i'_1, i'_2, \ldots, i'_n)$. Suppose $map(i_1, \ldots, i_n) = (a_1, b_1, \ldots, a_n, b_n)$ and $map(i'_1, \ldots, i'_n) = (a'_1, b'_1, \ldots, a'_n, b'_n)$.

We consider the following cases.
Case 1: $i_1 > i'_1$.
In this case
(1) $b_1 < b'_1$, and
(2) $a_1 = a'_1$.
From (1) it follows that
(3) $2b_1 < b'_1$.

Now, for $1 \leq i < n$, since $a_i < a_{i+1}$, and $b_i = h(-i_1 - a_i)$, we have $b_i > b_{i+1}$. Thus, $b_i > 2*b_{i+1}$, for $1 \leq i < n$ by hypothesis about elements of S. Thus, $\sum_{1\leq i\leq n} b_i \leq 2b_1$. Along with (3), we have that $b'_1 \geq \sum_{1\leq i\leq n} b_i$. This along with (2), Proposition 18, Proposition 24 and Proposition 31 gives $coSEMI\_HULL^{n,S}_{a_1,b_1,\ldots,a_n,b_n} \subseteq coSEMI\_HULL^{1,S}_{a'_1,b'_1} \subseteq coSEMI\_HULL^{n,S}_{a'_1,b'_1,\ldots,a'_n,b'_n}$.

Case 2: For some j, $1 < j \leq n$, $i_k = i'_k$, for $1 \leq k < j$, but $i_j > i'_j$.

In this case

(4) $a_i = a'_i$ and $b_i = b'_i$, for $1 \leq i < j$.

(5) $b_j < b'_j$, and

(6) $a_j > a'_j$.

From (5) it follows that

(7) $2 * b_j < b'_j$.

Now, for $1 \leq i < n$, since $a_i < a_{i+1}$, $b_i = h(-i_1 - a_i)$, we have $b_i > b_{i+1}$. Thus, $b_i > 2 * b_{i+1}$, for $1 \leq i < n$, by hypothesis about elements of S.

Thus, $\sum_{j \leq i \leq n} b_i \leq 2b_j$. This, along with (7) gives us that $b'_j > \sum_{j \leq i \leq n} b_i$. Thus using (6), Proposition 18, Proposition 24, and Proposition 31 we have $coSEMI\_HULL^{n,S}_{a_1,b_1,\ldots,a_n,b_n} \subseteq coSEMI\_HULL^{j,S}_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j} \subseteq coSEMI\_HULL^{n,S}_{a'_1,b'_1,\ldots,a'_n,b'_n}$.

Claim follows from above cases. □

We let $\Theta(X) = \bigcup_{\langle i_1,\ldots,i_n\rangle \in X} coSEMI\_HULL^n_{map(i_1,\ldots,i_n)}$.

Thus, it follows that $\Theta(L^Q_{i_1,i_2,\ldots,i_n}) = coSEMI\_HULL^{n,S}_{map(i_1,\ldots,i_n)}$.

Define Ψ as follows. If a sequence α of grammars converges to a grammar for $coSEMI\_HULL^{n,S}_{map(i_1,\ldots,i_n)}$, then $\Psi(\alpha)$ converges to a grammar for $L^Q_{i_1,i_2,\ldots,i_n}$.

It is now easy to verify that Θ and Ψ witness that $\mathcal{L}^Q \leq^{\mathbf{TxtEx}} coSEMI\_HULL^{n,S}$. ■

Note that upper and lower bounds for $coSEMI\_HULL^{n,S}$ given by Theorems 10, 11, and 12 do not match. The lower bound in Theorem 12 above is best possible (for Q-classes involving components from *BASIC*). However it is open whether the upper bound can be improved for general n. For $n = 1$, we do know that the the upper bound can be improved to show that $coSEMI\_HULL^{1,S} \leq^{\mathbf{TxtEx}} HALF$ (which is optimal by Theorem 12).

12 Open Hulls - Intersections of Semi-Hulls

Now consider the class of languages-"figures" that are intersections of *SEMI_HULL*s adjacent to the x-axis (that is with the first "break" point $(a_1, 0)$) and reverse *SEMI_HULL*s adjacent to the y-axis (with the first "break" point $(0, a'_1)$). These figures are the open hulls.

We give the formal definition below (preceded by the formal definition of the reverse *SEMI_HULL*s adjacent to the y-axis).

Definition 23 $REV\_SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} = \{(x,y) \mid (y,x) \in SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}\}$.
$REV\_SEMI\_HULL^{n,S} = \{REV\_SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} \mid SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} \in SEMI\_HULL^{n,S}\}$.

Definition 24
$OP\_HULL^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m} = SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} \cap REV\_SEMI\_HULL^m_{c_1,d_1,\ldots,c_m,d_m}$.
$OP\_HULL^{n,m,S} = \{OP\_HULL^{n,m,S}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m} \mid SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} \in SEMI\_HULL^{n,S}, REV\_SEMI\_HULL^n_{c_1,d_1,\ldots,c_m,d_m} \in REV\_SEMI\_HULL^{m,S}$, and $\sum_{1 \leq i \leq n} b_i < \frac{1}{\sum_{1 \leq i \leq m} d_i}\}$.

The latter condition, $\sum_{1 \leq i \leq n} b_i < \frac{1}{\sum_{1 \leq i \leq m} d_i}$, ensures that the languages in *OP_HULL*s are infinite, and thus the corresponding geometrical figures are *open* hulls (otherwise, the given classes will not be learnable).

The following proposition is an obvious corollary of the definition of *OP_HULL*s.

Proposition 35 *Suppose $S \subseteq S'$. Then $OP\_HULL^{n,m,S} \leq^{\mathbf{TxtEx}} OP\_HULL^{n,m,S'}$.*

Surprisingly, unlike *SEMI_HULLs* and *coSEMI_HULL*, upper and lower bounds for *OP_HULLs* match. The following theorem establishes the lower bound for the *OP_HULLs*.

Theorem 13 *Suppose $n \geq 1$, $m \geq 1$. Let $Q = (q_1, \ldots, q_n)$, where each $q_i = INIT$. Then, (a) $\mathcal{L}^Q \leq^{\mathbf{TxtEx}} OP\_HULL^{n,m,S}$, and (b) $\mathcal{L}^Q \leq^{\mathbf{TxtEx}} OP\_HULL^{m,n,S}$.*

PROOF. We only show part (a). Part (b) can be proved similarly.

The desired reduction works very similarly to the analogous reduction in Theorem 5: we just fix a *REV_SEMI_HULL* and try to reduce a language in $\mathcal{L}^Q$ to an *SEMI_HULL* part of the *OP_HULL*; the "slope" of the *REV_SEMI_HULL* provides a "starting point" for learning the first "slope" b_1, thus *HALF* being replaced with *INIT* in the first component of Q.

Now we proceed with the formal proof.

Without loss of generality (using Propositions 28 and 35) we can assume that, for each $b, b' \in S$, if $b < b'$, then $2b < b'$. Let h be an isomorphism from Z to S such that $h(i) < h(i+1)$.

Let $(c_1, d_1, \ldots, c_m, d_m)$ be S-valid such that $\sum_{1 \leq i \leq m} d_i < 1/h(1)$ (note that there clearly exist such $c_1, d_1, \ldots, c_m, d_m$).

Let $map(i_1, i_2, \ldots, i_n) = (a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m)$, where a_j, b_j, $1 \leq j \leq n$ are defined as follows. $a_1 = i_1 + 1$, $b_1 = h(-a_1)$. Suppose we have defined $a_1, b_1, \ldots, a_k, b_k$. Then let $A_{k+1} = \{x \in N \mid x > a_k, [\sum_{1 \leq i \leq k} b_i * (x \dot{-} a_i)] \in N\}$, and then let a_{k+1} to be the $(i_{k+1}+1)$-th least element in A_{k+1}. Let $b_{k+1} = h(-a_{k+1})$.

Note that, for all $(i_1, i_2, \ldots, i_n) \in N^n$, if $map(i_1, i_2, \ldots, i_n) = (a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m)$, then since $a_i < a_{i+1}$, by definition of b_i, we have $b_i = h(-a_i) > h(-a_{i+1}) = b_{i+1}$. This along with requirement on S gives $b_i > 2b_{i+1}$. Thus, $\sum_{1 \leq i \leq n} b_i \leq 2b_1 \leq 2h(-a_1) \leq 2h(0) < h(1)$. Since, $\sum_{1 \leq i \leq m} d_i < 1/h(1)$, we immediately have that $\sum_{1 \leq i \leq n} b_i < \frac{1}{\sum_{1 \leq i \leq m} d_i}$, for all $(i_1, i_2, \ldots, i_n)$.

Claim 7 $(i_1, i_2, \ldots, i_n) <_Q (i'_1, i'_2, \ldots, i'_n)$ *implies* $OP\_HULL^{n,m,S}_{map(i_1, \ldots, i_n)} \subset OP\_HULL^{n,m,S}_{map(i'_1, \ldots, i'_n)}$.

PROOF. Suppose $(i_1, i_2, \ldots, i_n) <_Q (i'_1, i'_2, \ldots, i'_n)$. Suppose $map(i_1, \ldots, i_n) = (a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m)$ and $map(i'_1, \ldots, i'_n) = (a'_1, b'_1, \ldots, a'_n, b'_n; c_1, d_1, \ldots, c_m, d_m)$.

Let j be the least number such that $i_k = i'_k$, for $1 \leq k < j$, and $i_j \neq i'_j$.

Thus,

(1) $a_i = a'_i$ and $b_i = b'_i$, for $1 \leq i < j$.

(2) $b_j > b'_j$, and

(3) $a_j < a'_j$.

From (2) it follows that

(4) $b_j > 2 * b'_j$.

Now, for $1 \leq i < n$, since $a'_i < a'_{i+1}$, we have $b'_i = h(-a'_i) > h(-a'_{i+1}) = b'_{i+1}$. Thus, $b'_i > 2 * b'_{i+1}$, for $1 \leq i < n$ by hypothesis about elements of S.

Thus, $\sum_{j \leq i \leq n} b'_i \leq 2b'_j$. This, along with (4) gives us that $b_j > \sum_{j \leq i \leq n} b'_i$. Thus using (3), Proposition 18 and Proposition 24 we have $SEMI\_HULL^{n,S}_{a_1,b_1,\ldots,a_n,b_n} \subset SEMI\_HULL^{j,S}_{a_1,b_1,\ldots,a_j,b_j} \subseteq SEMI\_HULL^{n,S}_{a'_1,b'_1,\ldots,a'_n,b'_n}$.

Claim follows. □

We now define $\Theta(X)$ for any finite set X as follows.

We let $\Theta(X) = \bigcup_{\langle i_1, \ldots, i_n \rangle \in X} OP\_HULL^{n,m}_{map(i_1, \ldots, i_n)}$.

Thus, it follows that $\Theta(L^Q_{i_1,i_2,\ldots,i_n}) = OP\_HULL^{n,m,S}_{map(i_1,\ldots,i_n)}$.

Define Ψ as follows. If a sequence α of grammars converges to a grammar for $OP\_HULL^{n,m,S}_{map(i_1,\ldots,i_n)}$, then $\Psi(\alpha)$ converges to a grammar for $L^Q_{i_1,i_2,\ldots,i_n}$.

It is now easy to verify that Θ and Ψ witness that $\mathcal{L}^Q \leq^{\mathbf{TxtEx}} OP\_HULL^{n,m,S}$. ∎

Before proving our next theorem we need some propositions.

Definition 25 *Suppose* $(a_1, b_1, \ldots, a_j, b_j)$ *is valid.* $REV\_INTER(a_1, b_1, \ldots, a_j, b_j) = \{(x, y) \mid (y, x) \in INTER(a_1, b_1, \ldots, a_j, b_j)\}$.

Definition 26 Suppose $(a_1, b_1, \ldots, a_j, b_j)$ and $(c_1, d_1, \ldots, c_k, d_k)$ are valid. Then, $INT\_OP\_HULL(a_1, b_1, \ldots, a_j, b_j; c_1, d_1, \ldots, c_k, d_k) = \bigcap\{OP\_HULL^{n,m}_{a_1,b_1,\ldots,a_j,b_j,\ldots,a_n,b_n;c_1,d_1,\ldots,c_k,d_k,\ldots,c_m,d_m} \mid n \geq j$ and $m \geq k$ and $(a_1, b_1, \ldots, a_n, b_n)$ and $(c_1, d_1, \ldots, c_m, d_m)$ are valid and $\sum_{1\leq i\leq n} b_i < \frac{1}{\sum_{1\leq i\leq m} d_i}\}$.

Proposition 36 *Suppose* $(a_1, b_1, \ldots, a_j, b_j)$ *and* $(c_1, d_1, \ldots, c_k, d_k)$ *are valid, and* $\sum_{1\leq i\leq j} b_j < \frac{1}{\sum_{1\leq i\leq k} d_i}$. *Then,*

(a) $\{(x, y) \mid x \leq maxinter(a_1, b_1, \ldots, a_j, b_j), y = \min(\{y' \mid (x, y') \in INTER(a_1, b_1, \ldots, a_j, b_j)\})\} \subseteq INT\_OP\_HULL(a_1, b_1, \ldots, a_j, b_j; c_1, d_1, \ldots, c_k, d_k)$.

(b) $\{(x, y) \mid y \leq maxinter(c_1, d_1, \ldots, c_k, d_k), x = \min(\{x' \mid (x', y) \in REV\_INTER(c_1, d_1, \ldots, c_j, d_j)\})\} \subseteq INT\_OP\_HULL(a_1, b_1, \ldots, a_j, b_j; c_1, d_1, \ldots, c_k, d_k)$.

Proof. We show only part (a). Part (b) can be proved similarly. Suppose $(x_0, y_0) \in \{(x, y) \mid x \leq maxinter(a_1, b_1, \ldots, a_j, b_j), y = \min(\{y' \mid (x, y') \in INTER(a_1, b_1, \ldots, a_j, b_j)\})\}$. Then, $1 + \sum_{1\leq i\leq j} b_i(x_0 \dot{-} a_i) > y_0 \geq \sum_{1\leq i\leq j} b_i(x_0 \dot{-} a_i)$. Thus, $1 + \sum_{1\leq i\leq j} b_i * x_0 \geq y_0$. Hence,

(1) $\sum_{1\leq i\leq j} b_i \geq (y_0 - 1)/x_0$.

Clearly, $(x_0, y_0) \in INTER(a_1, b_1, \ldots, a_j, b_j)$. Thus, $(x_0, y_0) \in SEMI\_HULL^n_{a_1,b_1,\ldots,a_j,b_j,\ldots,a_n,b_n}$, for all valid $(a_1, b_1, \ldots, a_j, b_j, \ldots, a_n, b_n)$ (by definition of $INTER$).

Thus, if $(x_0, y_0) \notin INT\_OP\_HULL(a_1, b_1, \ldots, a_j, b_j; c_1, d_1, \ldots, c_k, d_k)$, there must exist a valid $(c_1, d_1, \ldots, c_k, d_k, \ldots, c_m, d_m)$, and $\sum_{1\leq i\leq j} b_i < \frac{1}{\sum_{1\leq i\leq m} d_i}$, such that $(x_0, y_0) \notin REV\_SEMI\_HULL^m_{c_1,d_1,\ldots,c_k,d_k,\ldots,c_m,d_m}$. But this would mean, $x_0 < \sum_{1\leq i\leq m} d_i(y_0 \dot{-} c_i) \leq \sum_{1\leq i\leq m} d_i(y_0 \dot{-} 1)$. Thus,

(2) $\sum_{1\leq i\leq m} d_i > x_0/(y_0 - 1)$.

From (1) and (2) we have

$\sum_{1\leq i\leq n} b_i > \frac{1}{\sum_{1\leq i\leq m} d_i}$. A contradiction to the hypothesis. ∎

Proposition 37

Suppose $(a_1, b_1, \ldots, a_j, b_j)$, $(a'_1, b'_1, \ldots, a'_j, b'_j)$, $(c_1, d_1, \ldots, c_k, d_k)$, $(c'_1, d'_1, \ldots, c'_k, d'_k)$, *are valid. Suppose further that* $\sum_{1\leq i\leq j} b_i < \frac{1}{\sum_{1\leq i\leq k} d_i}$, *and* $\sum_{1\leq i\leq j} b'_i < \frac{1}{\sum_{1\leq i\leq k} d'_i}$.

If $INT\_OP\_HULL(a'_1, b'_1, \ldots, a'_j, b'_j; c'_1, d'_1, \ldots, c'_k, d'_k) \subseteq OP\_HULL^{j,k}_{a_1,b_1,\ldots,a_j,b_j;c_1,d_1,\ldots,c_k,d_k}$, *then,* $INTER(a'_1, b'_1, \ldots, a'_j, b'_j) \subseteq SEMI\_HULL^j_{a_1,b_1,\ldots,a_j,b_j}$ *and* $REV\_INTER(c'_1, d'_1, \ldots, c'_k, d'_k) \subseteq REV\_SEMI\_HULL^k_{c_1,d_1,\ldots,c_k,d_k}$.

Proof. Suppose
$INT\_OP\_HULL(a'_1, b'_1, \ldots, a'_j, b'_j; c'_1, d'_1, \ldots, c'_k, d'_k) \subseteq OP\_HULL^{j,k}_{a_1,b_1,\ldots,a_j,b_j;c_1,d_1,\ldots,c_k,d_k}$.

Thus (using Proposition 36) $\{(x,y) \mid x \leq \mathit{maxinter}(a_1', b_1', \ldots, a_j', b_j'), y = \min(\{y' \mid (x,y') \in \mathit{INTER}(a_1', b_1', \ldots, a_j', b_j')\})\} \subseteq \mathit{INT\_OP\_HULL}(a_1', b_1', \ldots, a_j', b_j'; c_1', d_1', \ldots, c_k', d_k') \subseteq \mathit{OP\_HULL}_{a_1,b_1,\ldots,a_j,b_j;c_1,d_1,\ldots,c_k,d_k} \subseteq \mathit{SEMI\_HULL}^j_{a_1,b_1,\ldots,a_j,b_j}$. It follows that $\mathit{INTER}(a_1', b_1', \ldots, a_j', b_j') \subseteq \mathit{SEMI\_HULL}^j_{a_1,b_1,\ldots,a_j,b_j}$.

Similarly, it can be shown that $\mathit{REV\_INTER}(c_1', d_1', \ldots, c_k', d_k') \subseteq \mathit{REV\_SEMI\_HULL}^k_{c_1,d_1,\ldots,c_k,d_k}$. ∎

Corollary 11 *Suppose $1 \leq j \leq n$, $1 \leq k \leq n$. Let S be any* **rat**$^+$*-covering set.*

Suppose $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$, $(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k, d_k)$ are S-valid, and $\sum_{1 \leq i \leq j} b_i < \frac{1}{\sum_{1 \leq i \leq k} d_i}$.

Then, there exist only finitely many (a_j', b_j', c_k', d_k') such that

(i) $b_j' + \sum_{1 \leq i < j} b_i < \frac{1}{d_k' + \sum_{1 \leq i < k} d_k}$, and

(ii) $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j', b_j')$, $(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k', d_k')$ are S-valid, and

(iii) $\mathit{INTER}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j', b_j') \subseteq \mathit{SEMI\_HULL}^j_{a_1,b_1,\ldots,a_j,b_j}$, and $\mathit{REV\_INTER}(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k', d_k') \subseteq \mathit{REV\_SEMI\_HULL}^k_{c_1,d_1,\ldots,c_k,d_k}$.

Moreover, canonical index for the finite set of (a_j', b_j', c_k', d_k') satisfying above three conditions can be obtained effectively from $a_1, b_1, \ldots, a_j, b_j, c_1, d_1, \ldots, c_k, d_k$.

Furthermore, for any (a_j', b_j', c_j', d_j') satisfying the above three conditions, $a_j' \leq a_j$, $c_k' \leq c_k$, and if $a_j' = a_j$ then $b_j' \geq b_j$, and if $c_k' = c_k$, then $d_k' \geq d_k$.

PROOF. Suppose $\mathit{INTER}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j', b_j') \subseteq \mathit{SEMI\_HULL}^j_{a_1,b_1,\ldots,a_j,b_j}$, and $\mathit{REV\_INTER}(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k', d_k') \subseteq \mathit{REV\_SEMI\_HULL}^k_{c_1,d_1,\ldots,c_k,d_k}$. By Corollary 5 and Proposition 23 it follows that $a_j' \leq a_j$, $c_k' \leq c_k$, and there exists only finitely many (a_j', b_j') such that $b_j' \leq b_j$ and $\mathit{INTER}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j', b_j') \subseteq \mathit{SEMI\_HULL}^j_{a_1,b_1,\ldots,a_j,b_j}$, and only finitely many (c_k', d_k') such that $d_k' \leq d_k$ and $\mathit{REV\_INTER}(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k', d_k') \subseteq \mathit{REV\_SEMI\_HULL}^k_{c_1,d_1,\ldots,c_k,d_k}$, and these (a_j', b_j'), (c_k', d_k') can be obtained effectively from $(a_1, b_1, \ldots, a_j, b_j; c_1, d_1, \ldots, c_k, d_k)$. Let $B_j' = \min(\{b_j\} \cup \{b_j' \mid b_j' \leq b_j \wedge \mathit{INTER}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j', b_j') \subseteq \mathit{SEMI\_HULL}^j_{a_1,b_1,\ldots,a_j,b_j}\})$. Let $D_k' = \min(\{d_k\} \cup \{d_k' \mid d_k' \leq d_k \wedge \mathit{REV\_INTER}(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k', d_k') \subseteq \mathit{REV\_SEMI\_HULL}^k_{c_1,d_1,\ldots,c_k,d_k}\})$.

Note that clause (i) in the Corollary implies that $b_j' < \frac{1}{d_k'}$ and $d_k' < \frac{1}{b_j'}$.

It follows that, for any (a_j', b_j', c_k', d_k') to satisfy the hypothesis of the theorem we must have $0 \leq a_j' \leq a_j$, $0 \leq c_k' \leq c_k$, $B_j' \leq b_j' \leq 1/D_k'$, and $D_k' \leq d_k' \leq 1/B_j'$. Thus, there exist only finitely many (a_j', b_j', c_k', d_k') which can satisfy clauses (i), (ii) and (iii) of the Corollary. Moreover since for any (a_j', b_j', c_k', d_k') it is effectively testable whether clauses (i), (ii) and (iii) of the Corollary are satisfiable, we can find the canonical index for the set of (a_j', b_j', c_k', d_k') satisfying the clauses (i), (ii) and (iii) of the corollary effectively from $a_1, b_1, \ldots, a_j, b_j, c_1, d_1, \ldots, c_k, d_k$.

Furthermore clause of the corollary follows using Corollary 5. ∎

Proposition 38 *Let S be any* **rat**$^+$*-covering set. Then, there exists a recursive function* Icode *with domain $\mathit{VALID}_S \times \mathit{VALID}_S$, and range $\subseteq N$ such that following is satisfied.*

Suppose $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$, $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j', b_j')$, $(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k, d_k)$, $(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k', d_k')$ are S-valid, and $\sum_{1 \leq i \leq j} b_i < \frac{1}{\sum_{1 \leq i \leq k} d_i}$ and $b_j' + \sum_{1 \leq i < j} b_i < \frac{1}{d_k' + \sum_{1 \leq i < k} d_i}$. Then

(A) If $\mathit{INTER}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j', b_j') \subseteq \mathit{SEMI\_HULL}^j_{a_1,b_1,\ldots,a_j,b_j}$, and $\mathit{REV\_INTER}(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k', d_k') \subseteq \mathit{REV\_SEMI\_HULL}^k_{c_1,d_1,\ldots,c_k,d_k}$ then

$\mathrm{Icode}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k) \leq \mathrm{Icode}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k, d_k)$.

(B) If $(a_j, b_j, c_k, d_k) \neq (a'_j, b'_j, c'_k, d'_k)$, *then* $\mathrm{Icode}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k) \neq \mathrm{Icode}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k, d_k)$.

(C) $\{\mathrm{Icode}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a''_j, b''_j; c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c''_k, d''_k) \mid (a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a''_j, b''_j) \in VALID_S, (c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c''_j, d''_j) \in VALID_S$, *and* $b''_j + \sum_{1 \leq i < j} b_j < \frac{1}{d''_k + \sum_{1 \leq i < k} d_k}\} = N$.

Proof can be done along similar lines as of Proposition 27.

Now we show the upper bound for *OP_HULL*s.

Theorem 14 *Let S be any* $\mathbf{rat}^+$*-covering set. Suppose* $n \geq m \geq 1$*. Let* $Q = (q_1, \ldots, q_n)$*, where each* $q_i = INIT$*. Then, (a)* $OP\_HULL^{n,m,S} \leq^{\mathbf{TxtEx}} \mathcal{L}^Q$*, and (b)* $OP\_HULL^{m,n,S} \leq^{\mathbf{TxtEx}} \mathcal{L}^Q$*.*

Proof. We show only part (a). Part (b) can be done similarly.

Intuitively, we use *INIT* type strategy to learn evey set of parameters (a_i, b_i, c_i, d_i). This is possible based on Proposition 38 above.

Now we proceed with the formal proof.

Let h be recursive bijection from Z to S such that $h(i) < h(i+1)$, for $i \in Z$. Let Icode be as in Proposition 38. For $(a_1, b_1, \ldots, a_n, b_n)$ and $(c_1, d_1, \ldots, c_m, d_m)$ in $VALID_S$, such that $\sum_{1 \leq i \leq n} b_i < \frac{1}{\sum_{1 \leq i \leq m} d_i}$, we let $map(a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m) = (\mathrm{Icode}(a_1, b_1; c_1, d_1), \mathrm{Icode}(a_1, b_1, a_2, b_2; c_1, d_1, c_2, d_2), \ldots, \mathrm{Icode}(a_1, b_1, \ldots, a_m, b_m; c_1, d_1, \ldots, c_m, d_m), \mathrm{Icode}(a_1, b_1, \ldots, a_{m+1}, b_{m+1}; c_1, d_1, \ldots, c_m, d_m), \ldots \mathrm{Icode}(a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m))$

Claim 8 *Suppose* $(a_1, b_1, \ldots, a_n, b_n)$ *and* $(a'_1, b'_1, \ldots, a'_n, b'_n)$, $(c_1, d_1, \ldots, c_m, d_m)$ *and* $(c'_1, d'_1, \ldots, c'_m, d'_m)$*, are S-valid, and* $\sum_{1 \leq i \leq n} b_i < \frac{1}{\sum_{1 \leq i \leq m} d_i}$*, and* $\sum_{1 \leq i \leq n} b'_i < \frac{1}{\sum_{1 \leq i \leq m} d'_i}$*.*

(A) Suppose $map(a'_1, b'_1, \ldots, a'_n, b'_n; c'_1, d'_1, \ldots, c'_m, d'_m) <_Q map(a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m)$*. Then, for the least j such that* $(a_j, b_j, c_{\min(j,m)}, d_{\min(j,m)}) \neq (a'_j, b'_j, c'_{\min(j,m)}, d'_{\min(j,m)})$*,* $INTER(a_1, b_1, \ldots, a_j, b_j) \not\subseteq SEMI\_HULL^j_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j}$*, or* $REV\_INTER(c_1, d_1, \ldots, c_{\min(j,m)}, d_{\min(j,m)}) \not\subseteq REV\_SEMI\_HULL^j_{c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)}}$*.*

(B) Suppose $map(a'_1, b'_1, \ldots, a'_n, b'_n; c'_1, d'_1, \ldots, c'_m, d'_m) <_Q map(a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m)$*. Then,* $INTER(a_1, b_1, \ldots, a_j, b_j, \ldots, a_n, b_n) \not\subseteq SEMI\_HULL^n_{a'_1, b'_1, \ldots, a'_n, b'_n}$ *or* $REV\_INTER(c_1, d_1, \ldots, c_m, d_m) \not\subseteq REV\_SEMI\_HULL^m_{c'_1, d'_1, \ldots, c'_m, d'_m}$*.*

(C) Suppose $OP\_HULL^{n,m}_{a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m} \subset OP\_HULL^{n,m}_{a'_1, b'_1, \ldots, a'_n, b'_n; c'_1, d'_1, \ldots, c'_m, d'_m}$*. Then* $map(a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m) <_Q map(a'_1, b'_1, \ldots, a'_n, b'_n; c'_1, d'_1, \ldots, c'_m, d'_m)$*.*

(D) Suppose $map(a'_1, b'_1, \ldots, a'_n, b'_n; c'_1, d'_1, \ldots, c'_m, d'_m) <_Q map(a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m)$*. Then,* $\{(x, y) \in N^2 \mid x \leq maxinter(a_1, b_1, \ldots, a_n, b_n)$ *and* $y = \min(\{y' \mid (x, y') \in INTER(a_1, b_1, \ldots, a_n, b_n)\})$ *OR* $y \leq maxinter(c_1, d_1, \ldots, c_m, d_m)$ *and* $x = \min(\{x' \mid (x', y) \in REV\_INTER(c_1, d_1, \ldots, c_m, d_m)\})\} \not\subseteq OP\_HULL^{n,m}_{a'_1, b'_1, \ldots, a'_n, b'_n; c'_1, d'_1, \ldots, c'_m, d'_m}$*.*

Proof. (A)
Let j be least number such that $(a_j, b_j, c_{\min(j,m)}, d_{\min(j,m)}) \neq (a'_j, b'_j, c'_{\min(j,m)}, d'_{\min(j,m)})$.

Note that, for $i < j$, we must have $\text{Icode}(a_1, b_1, \ldots, a_i, b_i; c_1, d_1, \ldots, c_{\min(i,m)}, d_{\min(i,m)}) = \text{Icode}(a'_1, b'_1, \ldots, a'_i, b'_i; c'_1, d'_1, \ldots, c'_{\min(i,m)}, d'_{\min(i,m)})$. If $\mathit{INTER}(a_1, b_1, \ldots, a_j, b_j) \subseteq \mathit{SEMI\_HULL}^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j}$ and $\mathit{REV\_INTER}(c_1, d_1, \ldots, c_{\min(j,m)}, d_{\min(j,m)}) \subseteq \mathit{REV\_SEMI\_HULL}^j_{c_1,d_1,\ldots,c_{\min(j,m)-1},d_{\min(j,m)-1},c'_{\min(j,m)},d'_{\min(j,m)}}$, then, by Proposition 38 we would have $\text{Icode}(a_1, b_1, \ldots, a_j, b_j; c_1, d_1, \ldots, c_{\min(j,m)}, d_{\min(j,m)}) < \text{Icode}(a'_1, b'_1, \ldots, a'_j, b'_j; c'_1, d'_1, \ldots, c'_{\min(j,m)}, d'_{\min(j,m)})$. Thus, $\mathit{map}(a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_{\min(j,m)}, d_{\min(j,m)}) <_Q \mathit{map}(a'_1, b'_1, \ldots, a'_n, b'_n; c'_1, d'_1, \ldots, c'_{\min(j,m)}, d'_{\min(j,m)})$, a contradiction to the hypothesis.

(B)

Follows from the fact that $\mathit{INTER}(a_1, b_1, \ldots, a_j, b_j) \subseteq \mathit{INTER}(a_1, b_1, \ldots, a_j, b_j, \ldots, a_n, b_n)$, $\mathit{REV\_INTER}(c_1, d_1, \ldots, c_{\min(j,m)}, d_{\min(j,m)}) \subseteq \mathit{REV\_INTER}(c_1, d_1, \ldots, c_{\min(j,m)}, d_{\min(j,m)}, \ldots, c_m, d_m)$, $\mathit{SEMI\_HULL}^n_{a'_1,b'_1,\ldots,a'_n,b'_n} \subseteq \mathit{SEMI\_HULL}^j_{a'_1,b'_1,\ldots,a'_j,b'_j}$ and $\mathit{REV\_SEMI\_HULL}^m_{c'_1,d'_1,\ldots,c'_m,d'_m} \subseteq \mathit{REV\_SEMI\_HULL}^j_{c'_1,d'_1,\ldots,c'_{\min(j,m)},d'_{\min(j,m)}}$ and part (A).

(C) If $\mathit{OP\_HULL}^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m} \subset \mathit{OP\_HULL}^{n,m}_{a'_1,b'_1,\ldots,a'_n,b'_n;c'_1,d'_1,\ldots,c'_m,d'_m}$, then by Proposition 37, $\mathit{INTER}(a_1, b_1, \ldots, a_n, b_n) \subseteq \mathit{SEMI\_HULL}^n_{a'_1,b'_1,\ldots,a'_n,b'_n}$ and $\mathit{REV\_INTER}(c_1, d_1, \ldots, c_m, d_m) \subseteq \mathit{REV\_SEMI\_HULL}^m_{c'_1,d'_1,\ldots,c'_m,d'_m}$. Now part (C) follows from part (B).

(D) Follows from (B) and definition of $\mathit{OP\_HULL}^{n,m}$. □

We now continue with the proof of the theorem. The aim is to construct Θ which maps $\mathit{OP\_HULL}^{n,m,S}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m}$ to $L^Q_{\mathit{map}(a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m)}$.

Note that definition of Ψ mapping grammar sequence converging to a grammar for $L^Q_{\mathit{map}(a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m)}$ to a grammar sequence converging to a grammar for $\mathit{OP\_HULL}^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m}$ would be trivial. We thus just define Θ.

Without loss of generality, we will be giving Θ as mapping sets to sets.

For any finite $X \subseteq N^2$, let $\mathit{Prop}(X, a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m)$ be true iff following two properties are satisfied.

(A) $(a_1, b_1, \ldots, a_n, b_n) \in \mathit{VALID}_S$, $(c_1, d_1, \ldots, c_m, d_m) \in \mathit{VALID}_S$, and $\sum_{1\leq i\leq n} b_i < \frac{1}{\sum_{1\leq i\leq m} d_i}$.

(B) For all $(a'_1, b'_1, \ldots, a'_n, b'_n), (c'_1, d'_1, \ldots, c'_m, d'_m) \in \mathit{VALID}_S$, $\sum_{1\leq i\leq n} b'_i < \frac{1}{\sum_{1\leq i\leq m} d'_i}$, such that $\mathit{map}(a'_1, b'_1, \ldots, a'_n, b'_n; c'_1, d'_1, \ldots, c'_m, d'_m) <_Q \mathit{map}(a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m)$, $X \not\subseteq \mathit{OP\_HULL}^{n,m}_{a'_1,b'_1,\ldots,a'_n,b'_n;c'_1,d'_1,\ldots,c'_m,d'_m}$.

Note that condition (B) above is equivalent to

(B') For all j, $1 \leq j \leq n$, for all $a'_j, c'_{\min(j,m)} \in N$, $b'_j, d'_{\min(j,m)} \in S$, such that $b'_j + \sum_{1\leq i<j} b_i < \frac{1}{d'_{\min(j,m)} + \sum_{1\leq i<\min(j,m)} d'_i}$,

if $\text{Icode}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)}) < \text{Icode}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c_{\min(j,m)}, d_{\min(j,m)})$, then

$[X \not\subseteq \mathit{OP\_HULL}^{j,\min(j,m)}_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j;c_1,d_1,\ldots,c_{\min(j,m)-1},d_{\min(j,m)-1},c'_{\min(j,m)},d'_{\min(j,m)}}]$.

Note that whether $X, a_1, b_1, \ldots, a_n, b_n, c_1, d_1, \ldots, c_m, d_m$, satisfy (A) and (B'), for all j, $1 \leq j \leq n$, is effectively testable.

Thus, for finite $X \subseteq N^2$, let

$\Theta(X) = \bigcup\{L^Q_{\mathit{map}(a_1,b_1,\ldots,a_n,b_n)} \mid \mathit{Prop}(X, a_1, b_1, \ldots, a_n, b_n)\}$.

For infinite X', $\Theta(X') = \bigcup_{X\subseteq X', \text{card}(X)<\infty} \Theta(X)$.

It is easy to verify that

(1) for any $X \subseteq OP\_HULL^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m}$,

$\Theta(X) \subseteq L^Q_{map(a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m)}$ (due to clause (B) in definition of *Prop* above, and the fact that for any valid I and I', $map(I) <_Q map(I')$, implies $L^Q_{map(I)} \subseteq L^Q_{map(I')}$), and

(2) For any finite set X such that $\{(x,y) \in N^2 \mid x \leq maxinter(a_1,b_1,\ldots,a_n,b_n)$ and $y = \min(\{y' \mid (x,y') \in INTER(a_1,b_1,\ldots,a_n,b_n)\})$ OR $y \leq maxinter(c_1,d_1,\ldots,c_m,d_m)$ and $x = \min(\{x' \mid (x',y) \in REV\_INTER(c_1,d_1,\ldots,c_m,d_m)\})\} \subseteq X \subseteq OP\_HULL^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m}$,

$\Theta(X) \supseteq L^Q_{map(a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m)}$.

(By Claim 8(D), and definition of Prop and Θ).

Thus, we have that $\Theta(OP\_HULL^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m}) = L^Q_{map(a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m)}$. ∎

13 Complements of Open Hulls

In this section we define and explore the classes of complements of *OP_HULL*s.

Definition 27
$REV\_coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} = \{(x,y) \mid (y,x) \in coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}\}$.
$REV\_coSEMI\_HULL^{n,S} = \{REV\_coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} \mid coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} \in coSEMI\_HULL^{n,S}\}$.

Definition 28 $coOP\_HULL^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m} = coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} \cup REV\_coSEMI\_HULL^m_{c_1,d_1,\ldots,c_m,d_m} = N^2 - OP\_HULL^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m}$.
$coOP\_HULL^{n,m,S} = \{coOP\_HULL^{n,m,S}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m} \mid coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} \in coSEMI\_HULL^{n,S}, REV\_coSEMI\_HULL^m_{c_1,d_1,\ldots,c_m,d_m} \in REV\_coSEMI\_HULL^{m,S}$, and $\sum_{1\leq i\leq n} b_i < \frac{1}{\sum_{1\leq i\leq m} d_i}\}$.

Proposition 39 *Suppose $S \subseteq S'$. Then $coOP\_HULL^{n,m,S} \leq^{\mathbf{TxtEx}} coOP\_HULL^{n,m,S'}$.*

The following theorem gives the lower bound for *coOP_HULL*s.

Theorem 15 *Suppose $n \geq 1$, $m \geq 1$. Let $Q = (q_1,\ldots,q_n)$, where each $q_i = COINIT$. Then, (a) $\mathcal{L}^Q \leq^{\mathbf{TxtEx}} coOP\_HULL^{n,m,S}$, and (b) $\mathcal{L}^Q \leq^{\mathbf{TxtEx}} coOP\_HULL^{m,n,S}$.*

Proof. We only show part (a). Part (b) can be proved similarly. The proof is very similar to the proof of lower bound for *OP_HULL*s with *INIT*s being replaced by *COINIT*s. Without loss of generality (using Propositions 28 and 39) we can assume that, for each $b, b' \in S$, if $b < b'$, then $2b < b'$. Let h be an isomorphism from Z to S such that $h(i) < h(i+1)$.

Let $(c_1,d_1,\ldots,c_m,d_m)$ be S-valid such that $\sum_{1\leq i\leq m} d_i < 1/h(1)$ (note that there clearly exist such $c_1,d_1,\ldots,c_m,d_m$).

Let $map(i_1,i_2,\ldots,i_n) = (a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m)$, where a_j, b_j, $1 \leq j \leq n$ are defined as follows. $a_1 = i_1+1$, $b_1 = h(-a_1)$. Suppose we have defined $a_1,b_1,\ldots,a_k,b_k$. Then let $A_{k+1} = \{x \in N \mid x > a_k, [\sum_{1\leq i\leq k} b_i * (x \dot{-} a_i)] \in N\}$, and then let a_{k+1} to be the $(i_{k+1}+1)$-th least element in A_{k+1}. Let $b_{k+1} = h(-a_{k+1})$.

Note that, for all $(i_1,i_2,\ldots,i_n) \in N^n$, if $map(i_1,i_2,\ldots,i_n) = (a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m)$, then since $a_i < a_{i+1}$, by definition of b_i, we have $b_i = h(-a_i) > h(-a_{i+1}) = b_{i+1}$. This along with requirement on S gives $b_i > 2b_{i+1}$. Thus, $\sum_{1\le i\le n} b_i \le 2b_1 \le 2h(-a_1) \le 2h(0) < h(1)$. Since, $\sum_{1\le i\le m} d_i < 1/h(1)$, we immediately have that $\sum_{1\le i\le n} b_i < \frac{1}{\sum_{1\le i\le m} d_i}$, for all $(i_1, i_2, \ldots, i_n)$.

Claim 9 $(i_1, i_2, \ldots, i_n) <_Q (i'_1, i'_2, \ldots, i'_n)$ *implies* $coOP\_HULL^{n,m,S}_{map(i_1,\ldots,i_n)} \subset coOP\_HULL^{n,m,S}_{map(i'_1,\ldots,i'_n)}$.

PROOF. Suppose $(i_1, i_2, \ldots, i_n) <_Q (i'_1, i'_2, \ldots, i'_n)$. Suppose $map(i_1, \ldots, i_n) = (a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m)$ and $map(i'_1, \ldots, i'_n) = (a'_1, b'_1, \ldots, a'_n, b'_n; c_1, d_1, \ldots, c_m, d_m)$.

Let j be the least number such that $i_k = i'_k$, for $1 \le k < j$, and $i_j \ne i'_j$.

Thus,

(1) $a_i = a'_i$ and $b_i = b'_i$, for $1 \le i < j$.

(2) $b_j < b'_j$, and

(3) $a_j > a'_j$.

From (2) it follows that

(4) $2b_j < b'_j$.

Now, for $1 \le i < n$, since $a_i < a_{i+1}$, we have $b_i = h(-a_i) > h(-a_{i+1}) = b_{i+1}$. Thus, $b_i > 2 * b_{i+1}$, for $1 \le i < n$ by hypothesis about elements of S.

Thus, $\sum_{j\le i\le n} b_i \le 2b_j$. This, along with (4) gives us that $b'_j > \sum_{j\le i\le n} b_i$. Thus using (3), Proposition 18, Proposition 24, and Proposition 31, we have $coSEMI\_HULL^{n,S}_{a_1,b_1,\ldots,a_n,b_n} \subset coSEMI\_HULL^{j,S}_{a'_1,b'_1,\ldots,a'_j,b'_j} \subseteq coSEMI\_HULL^{n,S}_{a'_1,b'_1,\ldots,a'_n,b'_n}$.

Claim follows. □

We now define $\Theta(X)$ for any finite set X as follows.

We let $\Theta(X) = \bigcup_{\langle i_1,i_2,\ldots,i_n\rangle\in X} coOP\_HULL^{n,m}_{map(i_1,\ldots,i_n)}$.

Thus, it follows that $\Theta(L^Q_{i_1,i_2,\ldots,i_n}) = coOP\_HULL^{n,m,S}_{map(i_1,\ldots,i_n)}$.

Define Ψ as follows. If a sequence α of grammars converges to a grammar for $coOP\_HULL^{n,m,S}_{map(i_1,\ldots,i_n)}$, then $\Psi(\alpha)$ converges to a grammar for $L^Q_{i_1,i_2,\ldots,i_n}$.

It is now easy to verify that Θ and Ψ witness that $\mathcal{L}^Q \le^{\mathbf{TxtEx}} coOP\_HULL^{n,m,S}$. ∎

Before proving our next theorem, the upper bound, we need some propositions.

Definition 29 *Suppose* $(a_1, b_1, \ldots, a_j, b_j)$ *is valid.* $REV\_coINTER(a_1, b_1, \ldots, a_j, b_j) = \{(x, y) \mid (y, x) \in coINTER(a_1, b_1, \ldots, a_j, b_j)\}$.

Definition 30 Suppose $(a_1, b_1, \ldots, a_j, b_j)$ and $(c_1, d_1, \ldots, c_k, d_k)$ are valid. Then, $coINT\_OP\_HULL(a_1, b_1, \ldots, a_j, b_j; c_1, d_1, \ldots, c_k, d_k) = \bigcup\{coOP\_HULL^{n,m}_{a_1,b_1,\ldots,a_j,b_j,\ldots,a_n,b_n;c_1,d_1,\ldots,c_k,d_k,\ldots,c_m,d_m} \mid n \ge j$ and $m \ge k$ and $(a_1, b_1, \ldots, a_n, b_n)$ and $(c_1, d_1, \ldots, c_m, d_m)$ are valid and $\sum_{1\le i\le n} b_i < \frac{1}{\sum_{1\le i\le m} d_i}\} = N^2 - INT\_OP\_HULL(a_1, b_1, \ldots, a_j, b_j; c_1, d_1, \ldots, c_k, d_k)$.

Proposition 40 *Suppose* $(a_1, b_1, \ldots, a_j, b_j)$ *and* $(c_1, d_1, \ldots, c_k, d_k)$ *are valid. Suppose* $\sum_{1\le i\le j} b_i < \frac{1}{\sum_{1\le i\le k} d_i}$. *Let* $(x, y) \in N^2$ *be such that* $1 + \sum_{1\le i<j} b_i(x \dot{-} a_i) \ge y \ge \sum_{1\le i<j} b_i(x \dot{-} a_i)$. *Then, if* $(x, y) \in coINT\_OP\_HULL(a_1, b_1, \ldots, a_j, b_j; c_1, d_1, \ldots, c_k, d_k)$, *then* $(x, y) \in coINTER(a_1, b_1, \ldots, a_j, b_j)$.

PROOF. Suppose $(x, y) \in coINT\_OP\_HULL(a_1, b_1, \ldots, a_j, b_j; c_1, d_1, \ldots, c_k, d_k)$. Then, $(x, y) \in \bigcup\{coOP\_HULL^{n,m}_{a_1,b_1,\ldots,a_j,b_j,\ldots,a_n,b_n;c_1,d_1,\ldots,c_k,d_k,\ldots,c_m,d_m} \mid n \ge j$ and $m \ge k$ and $(a_1, b_1, \ldots, a_n, b_n)$ and $(c_1, d_1, \ldots, c_m, d_m)$ are valid and $\sum_{1\le i\le n} b_i < \frac{1}{\sum_{1\le i\le m} d_i}\}$.

Thus $(x,y) \in \bigcup\{coSEMI\_HULL^n_{a_1,b_1,\ldots,a_j,b_j,\ldots,a_n,b_n} \mid n \geq j$ and $(a_1,b_1,\ldots,a_n,b_n)$ is valid$\} \cup \bigcup\{REV_coSEMI_HULL^m_{c_1,d_1,\ldots,c_k,d_k,\ldots,c_m,d_m} \mid m \geq k$ and $(c_1,d_1,\ldots,c_m,d_m)$ is valid and $\sum_{1\leq i\leq m} d_i < \frac{1}{\sum_{1\leq i<j} b_i}\}$. Thus, $(x,y) \in coINTER(a_1,b_1,\ldots,a_j,b_j) \cup \bigcup\{REV\_coSEMI\_HULL^m_{c_1,d_1,\ldots,c_k,d_k,\ldots,c_m,d_m} \mid m \geq k$ and $(c_1,d_1,\ldots,c_m,d_m)$ is valid and $\sum_{1\leq i\leq m} d_i < \frac{1}{\sum_{1\leq i<j} b_i}\}$.

We claim that $(x,y) \notin \bigcup\{REV\_coSEMI\_HULL^m_{c_1,d_1,\ldots,c_k,d_k,\ldots,c_m,d_m} \mid m \geq k$ and $(c_1,d_1,\ldots,c_m,d_m)$ is valid and $\sum_{1\leq i\leq m} d_i < \frac{1}{\sum_{1\leq i<j} b_i}\}$. This would prove the proposition. Suppose by way of contradiction that $(x,y) \in REV\_coSEMI\_HULL^m_{c_1,d_1,\ldots,c_k,d_k,\ldots,c_m,d_m}$, where $m \geq k$, $(c_1,d_1,\ldots,c_m,d_m)$ is valid and $\sum_{1\leq i\leq m} d_i < \frac{1}{\sum_{1\leq i<j} b_i}$. But, then $(x,y) \in REV\_coSEMI\_HULL^m_{c_1,\sum_{1\leq i\leq m} d_i}$. Thus, $x < \sum_{1\leq i\leq m} d_i(y \dot{-} 1)$. If $y \leq 1$, then clearly, above cannot happen. So assume $y > 1$. Thus, $x < \sum_{1\leq i\leq m} d_i(y-1)$. Hence

(1) $x/(y-1) < \sum_{1\leq i\leq m} d_i$.

However, $y \leq 1 + \sum_{1\leq i<j} b_i(x \dot{-} a_i)$. Thus, $y-1 \leq \sum_{1\leq i<j} b_i(x \dot{-} a_i)$. Thus, $y-1 \leq \sum_{1\leq i<j} b_i x$. Thus,

(2) $(y-1)/x \leq \sum_{1\leq i<j} b_i$.

Multiplying (1) and (2) we have $1 \leq [\sum_{1\leq i<j} b_i] * [\sum_{1\leq i\leq m} d_i]$. But then, $\sum_{1\leq i\leq n} b_i \geq \sum_{1\leq i<j} b_i \geq 1/[\sum_{1\leq i\leq m} d_i]$. A contradiction. ∎

Similarly, one can show

Proposition 41 *Suppose $(a_1,b_1,\ldots,a_j,b_j)$ and $(c_1,d_1,\ldots,c_k,d_k)$ are valid. Suppose $\sum_{1\leq i\leq j} b_i < \frac{1}{\sum_{1\leq i\leq k} d_i}$. Let $(x,y) \in N^2$ be such that $1 + \sum_{1\leq i<j} d_i(y \dot{-} c_i) \geq x \geq \sum_{1\leq i<j} d_i(y \dot{-} c_i)$. Then, if $(x,y) \in coINT\_OP\_HULL(a_1,b_1,\ldots,a_j,b_j;c_1,d_1,\ldots,c_k,d_k)$, then $(x,y) \in REV\_coINTER(c_1,d_1,\ldots,c_k,d_k)$.*

Proposition 42
Suppose $(a_1,b_1,\ldots,a_j,b_j)$, $(a'_1,b'_1,\ldots,a'_j,b'_j)$, $(c_1,d_1,\ldots,c_k,d_k)$, $(c'_1,d'_1,\ldots,c'_k,d'_k)$, are valid. Suppose further that $\sum_{1\leq i\leq j} b_i < \frac{1}{\sum_{1\leq i\leq k} d_i}$, and $\sum_{1\leq i\leq j} b'_i < \frac{1}{\sum_{1\leq i\leq k} d'_i}$.

If $coINT\_OP\_HULL(a'_1,b'_1,\ldots,a'_j,b'_j;c'_1,d'_1,\ldots,c'_k,d'_k) \supseteq coOP\_HULL^{j,k}_{a_1,b_1,\ldots,a_j,b_j;c_1,d_1,\ldots,c_k,d_k}$, then, $coINTER(a'_1,b'_1,\ldots,a'_j,b'_j) \supseteq coSEMI\_HULL^j_{a_1,b_1,\ldots,a_j,b_j}$ and $REV\_coINTER(c'_1,d'_1,\ldots,c'_k,d'_k) \supseteq REV\_coSEMI\_HULL^k_{c_1,d_1,\ldots,c_k,d_k}$.

PROOF. Follows from Proposition 37. ∎

Theorem 16 *Suppose $n \geq m \geq 1$. Let $Q = (q_1,\ldots,q_n)$, where each $q_i = COINIT$. Then, (a) $coOP\_HULL^{n,m,S} \leq^{\mathbf{TxtEx}} \mathcal{L}^Q$, and (b) $coOP\_HULL^{m,n,S} \leq^{\mathbf{TxtEx}} \mathcal{L}^Q$.*

PROOF. We show only part (a). Part (b) can be done similarly.

Intuitively, we use *COINIT* type strategy to learn evey set of parameters (a_i,b_i,c_i,d_i). This is possible based on Proposition 38 above using a method similar to method as that in Theorem 14, though technical details become more complicated.

Let h be recursive bijection from Z to S such that $h(i) < h(i+1)$, for $i \in Z$. Let Icode be as in Proposition 38. For $(a_1,b_1,\ldots,a_n,b_n)$ and $(c_1,d_1,\ldots,c_m,d_m)$ in $VALID_S$, such that $\sum_{1\leq i\leq n} b_i < \frac{1}{\sum_{1\leq i\leq m} d_i}$, we let $map(a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m) = (\mathrm{Icode}(a_1,b_1;c_1,d_1)$, $\mathrm{Icode}(a_1,b_1,a_2,b_2;c_1,d_1,c_2,d_2)$, $\ldots$,

$\mathrm{Icode}(a_1, b_1, \ldots, a_m, b_m; c_1, d_1, \ldots, c_m, d_m)$, $\mathrm{Icode}(a_1, b_1, \ldots, a_{m+1}, b_{m+1}; c_1, d_1, \ldots, c_m, d_m)$, ... $\mathrm{Icode}(a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m))$

Claim 10 *Suppose* $(a_1, b_1, \ldots, a_j, b_j)$, $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$, $(c_1, d_1, \ldots, c_k, d_k)$, $(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k)$, *are S-valid, where* $\sum_{1 \le i \le j} b_j < \frac{1}{\sum_{1 \le i \le k} d_k}$, *and* $b'_j + \sum_{1 \le i < j} b_j < \frac{1}{d'_k + \sum_{1 \le i < k} d_k}$. *Suppose* $(a_j, b_j, c_k, d_k) \neq (a'_j, b'_j, c'_k, d'_k)$. *Suppose* $coINTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \supseteq coSEMI\_HULL^j_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j}$ *and* $REV\_coINTER(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k) \supseteq REV\_coSEMI\_HULL^k_{c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k, d_k}$, *then* $\mathrm{Icode}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k) < \mathrm{Icode}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k, d_k)$.

PROOF. By hypothesis, we have $INTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq SEMI\_HULL^j_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j}$ and $REV\_INTER(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k) \subseteq REV\_SEMI\_HULL^k_{c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k, d_k}$. Thus, we have $\mathrm{Icode}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k) < \mathrm{Icode}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k, d_k)$ (by Proposition 38; for getting $<$, use the fact that $(a_j, b_j, c_k, d_k) \neq (a'_j, b'_j, c'_k, d'_k)$). □

Claim 11 *Suppose* $(a_1, b_1, \ldots, a_n, b_n)$, $(c_1, d_1, \ldots, c_m, d_m)$ *are S-valid, and* $\sum_{1 \le i \le n} b_i < \frac{1}{\sum_{1 \le i \le m} d_i}$.

(A) Suppose $1 \le j \le n$, $1 \le k \le m$. *Suppose* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ *and* $(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k)$ *are S-valid,* $b'_j + \sum_{1 \le i < j} b_i < \frac{1}{d'_k + \sum_{1 \le i < k} d_i}$.

Suppose $\mathrm{Icode}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k) > \mathrm{Icode}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k, d_k)$. *Then* $[coINTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \not\supseteq coSEMI\_HULL^j_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j}$ *or* $REV\_coINTER(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k) \not\supseteq REV\_coSEMI\_HULL^k_{c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k, d_k}]$. *Thus,* $coINT\_OP\_HULL(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k) \not\supseteq coOP\_HULL^{j,k}_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k, d_k}$.

(B) Suppose $(a'_1, b'_1, \ldots, a'_n, b'_n)$ *and* $(c'_1, d'_1, \ldots, c'_m, d'_m)$ *are S-valid,* $\sum_{1 \le i \le n} b'_i < \frac{1}{\sum_{1 \le i \le m} d'_i}$. *Suppose* $map(a'_1, b'_1, \ldots, a'_n, b'_n; c'_1, d'_1, \ldots, c'_m, d'_m) <_Q map(a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m)$. *Then, for the least* j *such that* $(a_j, b_j, c_{\min(j,m)}, d_{\min(j,m)}) \neq (a'_j, b'_j, c'_{\min(j,m)}, d'_{\min(j,m)})$, $coINTER(a_1, b_1, \ldots, a'_j, b'_j) \not\supseteq coSEMI\_HULL^j_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j}$, *or* $REV\_coINTER(c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)}) \not\supseteq REV\_coSEMI\_HULL^j_{c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c_{\min(j,m)}, d_{\min(j,m)}}$.

(C) Suppose $(a'_1, b'_1, \ldots, a'_n, b'_n)$ *and* $(c'_1, d'_1, \ldots, c'_m, d'_m)$ *are S-valid,* $\sum_{1 \le i \le n} b'_i < \frac{1}{\sum_{1 \le i \le m} d'_i}$. *Suppose* $map(a'_1, b'_1, \ldots, a'_n, b'_n; c'_1, d'_1, \ldots, c'_m, d'_m) <_Q map(a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m)$. *Then,* $coINTER(a'_1, b'_1, \ldots, a'_j, b'_j, \ldots, a'_n, b'_n) \not\supseteq coSEMI\_HULL^n_{a_1, b_1, \ldots, a_n, b_n}$ *or* $REV\_coINTER(c'_1, d'_1, \ldots, c'_m, d'_m) \not\supseteq REV\_coSEMI\_HULL^m_{c_1, d_1, \ldots, c_m, d_m}$.

(D) Suppose $(a'_1, b'_1, \ldots, a'_n, b'_n)$ *and* $(c'_1, d'_1, \ldots, c'_m, d'_m)$ *are S-valid,* $\sum_{1 \le i \le n} b'_i < \frac{1}{\sum_{1 \le i \le m} d'_i}$. *Suppose* $coOP\_HULL^{n,m}_{a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m} \subset coOP\_HULL^{n,m}_{a'_1, b'_1, \ldots, a'_n, b'_n; c'_1, d'_1, \ldots, c'_m, d'_m}$. *Then* $map(a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m) <_Q map(a'_1, b'_1, \ldots, a'_n, b'_n; c'_1, d'_1, \ldots, c'_m, d'_m)$.

(E) For $1 \le j \le n$, *there exists a finite* $X_j \subseteq coOP\_HULL^{n,m}_{a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m}$ *such that, for all S-valid* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$, $(c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)})$, *such*

that $b'_j + \sum_{1\le i<j} b_i < \frac{1}{d'_{\min(j,m)}+\sum_{1\le i<\min(j,m)} d_i}$, *if* $\text{Icode}(a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j;c_1,d_1,\ldots,c_{\min(j,m)-1},d_{\min(j,m)-1},c'_{\min(j,m)},d'_{\min(j,m)}) > \text{Icode}(a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j;c_1,d_1,\ldots,c_{\min(j,m)-1},d_{\min(j,m)-1},c_{\min(j,m)},d_{\min(j,m)})$, *then,* $coINT\_OP\_HULL(a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j;c_1,d_1,\ldots,c_{\min(j,m)-1},d_{\min(j,m)-1},c'_{\min(j,m)},d'_{\min(j,m)}) \not\supseteq X_j$.

(F) There exists a finite $X \subseteq coOP\_HULL^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m}$ *such that, for all* j, $1\le j\le n$, *for S-valid* $(a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j)$, $(c_1,d_1,\ldots,c_{\min(j,m)-1},d_{\min(j,m)-1},c'_{\min(j,m)},d'_{\min(j,m)})$, *such that* $b'_j + \sum_{1\le i<j} b_i < \frac{1}{d'_{\min(j,m)}+\sum_{1\le i<\min(j,m)} d_i}$, *if* $\text{Icode}(a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j;c_1,d_1,\ldots,c_{\min(j,m)-1},d_{\min(j,m)-1},c'_{\min(j,m)},d'_{\min(j,m)}) > \text{Icode}(a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j;c_1,d_1,\ldots,c_{\min(j,m)-1},d_{\min(j,m)-1},c_{\min(j,m)},d_{\min(j,m)})$, *then,* $coINT\_OP\_HULL(a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j;c_1,d_1,\ldots,c_{\min(j,m)-1},d_{\min(j,m)-1},c'_{\min(j,m)},d'_{\min(j,m)}) \not\supseteq X$.

PROOF. (A) Suppose the hypothesis. Then, it follows from Claim 10 that $[coINTER(a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j) \not\supseteq coSEMI\_HULL^{j}_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$ or $REV\_coINTER(c_1,d_1,\ldots,c_{k-1},d_{k-1},c'_k,d'_k) \not\supseteq REV\_coSEMI\_HULL^{k}_{c_1,d_1,\ldots,c_{k-1},d_{k-1},c_k,d_k}]$. Thus, by Proposition 42, we have $coINT\_OP\_HULL(a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j;c_1,d_1,\ldots,c_{k-1},d_{k-1},c'_k,d'_k) \not\supseteq coOP\_HULL^{j,k}_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j;c_1,d_1,\ldots,c_{k-1},d_{k-1},c_k,d_k}$.

(B) Let j be least number such that $(a_j,b_j,c_{\min(j,m)},d_{\min(j,m)}) \neq (a'_j,b'_j,c'_{\min(j,m)},d'_{\min(j,m)})$. Note that, for $i<j$, we must have $\text{Icode}(a_1,b_1,\ldots,a_i,b_i;c_1,d_1,\ldots,c_{\min(i,m)},d_{\min(i,m)}) = \text{Icode}(a'_1,b'_1,\ldots,a'_i,b'_i;c'_1,d'_1,\ldots,c'_{\min(i,m)},d'_{\min(i,m)})$. If $coINTER(a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j) \supseteq coSEMI\_HULL^{j}_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$ and $REV\_coINTER(c_1,d_1,\ldots,c_{\min(j,m)-1},d_{\min(j,m)-1},c'_{\min(j,m)},d'_{\min(j,m)}) \supseteq REV\_coSEMI\_HULL^{j}_{c_1,d_1,\ldots,c_{\min(j,m)-1},d_{\min(j,m)-1},c_{\min(j,m)},d_{\min(j,m)}}$, then, by part (A) we would have $\text{Icode}(a_1,b_1,\ldots,a_j,b_j;c_1,d_1,\ldots,c_{\min(j,m)},d_{\min(j,m)}) > \text{Icode}(B,a'_1,b'_1,\ldots,a'_j,b'_j;c'_1,d'_1,\ldots,c'_{\min(j,m)},d'_{\min(j,m)})$. Thus, $map(a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_{\min(j,m)},d_{\min(j,m)}) <_Q map(a'_1,b'_1,\ldots,a'_n,b'_n;c'_1,d'_1,\ldots,c'_{\min(j,m)},d'_{\min(j,m)})$, a contradiction to the hypothesis.

(C) Follows from the fact that, for $1\le j\le n$, $coINTER(a_1,b_1,\ldots,a_j,b_j) \supseteq coINTER(a_1,b_1,\ldots,a_j,b_j,\ldots,a_n,b_n)$, $REV\_coINTER(c_1,d_1,\ldots,c_{\min(j,m)},d_{\min(j,m)}) \supseteq REV\_coINTER(c_1,d_1,\ldots,c_{\min(j,m)},d_{\min(j,m)},\ldots,c_m,d_m)$, $coSEMI\_HULL^{n}_{a'_1,b'_1,\ldots,a'_n,b'_n} \supseteq coSEMI\_HULL^{j}_{a'_1,b'_1,\ldots,a'_j,b'_j}$ and $REV\_coSEMI\_HULL^{m}_{c'_1,d'_1,\ldots,c'_m,d'_m} \supseteq REV\_coSEMI\_HULL^{j}_{c'_1,d'_1,\ldots,c'_{\min(j,m)},d'_{\min(j,m)}}$ and part (B).

(D) If $coOP\_HULL^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m} \subset coOP\_HULL^{n,m}_{a'_1,b'_1,\ldots,a'_n,b'_n;c'_1,d'_1,\ldots,c'_m,d'_m}$, then $coOP\_HULL^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m} \subset coINT\_OP\_HULL^{n,m}_{a'_1,b'_1,\ldots,a'_n,b'_n;c'_1,d'_1,\ldots,c'_m,d'_m}$. Thus, by Proposition 42, $coSEMI\_HULL(a_1,b_1,\ldots,a_n,b_n) \subseteq coINTER^{n}_{a'_1,b'_1,\ldots,a'_n,b'_n}$ and $REV\_coSEMI\_HULL(c_1,d_1,\ldots,c_m,d_m) \subseteq REV\_coINTER^{m}_{c'_1,d'_1,\ldots,c'_m,d'_m}$. Now part (D) follows from part (C).

(E) Let $(x_1,y_1) \in coSEMI\_HULL^{j}_{a_1,b_1,\ldots,a_j,b_j}$ be such that $1+\sum_{1\le i<j} b_i(x_1 \dot- a_i) \ge y_1 > \sum_{1\le i<j} b_i(x_1 \dot- a_i)$. Note that there exists such (x_1,y_1). By Proposition 40 if $(x_1,y_1) \in coINT\_OP\_HULL(a_1,b_1,\ldots,a_{j-1},b_{j-1},a''_j,b''_j;c_1,d_1,\ldots,c_{\min(j,m)-1},d_{\min(j,m)-1},c''_{\min(j,m)},d''_{\min(j,m)})$, then $(x_1,y_1) \in coINTER(a_1,b_1,\ldots,a_{j-1},b_{j-1},a''_j,b''_j)$. Thus, by Proposition 33, there exists a

$B' \in \mathbf{rat}^+$ such that if $(x_1, y_1) \in coINT\_OP\_HULL(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a''_j, b''_j; c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c''_{\min(j,m)}, d''_{\min(j,m)})$, then $a''_j < x_1$ and $b''_j > B'$.

Similarly, let $(x_2, y_2) \in REV\_coSEMI\_HULL^{\min(j,m)}_{c_1,d_1,\ldots,c_{\min(j,m)},d_{\min(j,m)}}$ be such that $1 + \sum_{1\leq i<\min(j,m)} b_i(y_2 \dot{-} a_i) \geq x_2 > \sum_{1\leq i<\min(j,m)} b_i(y_2 \dot{-} a_i)$. Note that there exists such (x_2, y_2). By Proposition 41 if $(x_2, y_2) \in coINT\_OP\_HULL(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a''_j, b''_j; c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c''_{\min(j,m)}, d''_{\min(j,m)})$, then $(x_2, y_2) \in REV\_coINTER(c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c''_{\min(j,m)}, d''_{\min(j,m)})$. Thus, by Proposition 33, there exists a $B'' \in \mathbf{rat}^+$ such that if $(x_2, y_2) \in coINT\_OP\_HULL(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a''_j, b''_j; c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c''_{\min(j,m)}, d''_{\min(j,m)})$, then $c''_{\min(j,m)} < x_2$ and $d''_{\min(j,m)} > B''$.

Thus, for (x_1, y_1) and (x_2, y_2) to be in $coINT\_OP\_HULL^{j,\min(j,m)}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)})$, where $b'_j + \sum_{1\leq i<j} b_i < \frac{1}{d'_{\min(j,m)} + \sum_{1\leq i<\min(j,m)} d_i}$, we must have $a'_j \leq x_1$, $b'_j \geq B'$, $c'_{\min(j,m)} \leq x_2$, $d'_{\min(j,m)} \geq B''$, and $b'_j \leq 1/B''$ and $d_{\min(j,m)} \leq 1/B'$. Since there are only finitely many such $(a'_j, b'_j, c'_{\min(j,m)}, d'_{\min(j,m)})$ with $b'_j, d'_{\min(j,m)} \in S$, we have part (E) using part (A).

(F) Let $X = \bigcup_{1\leq j\leq n} X_j$ where X_j is as in part (E). Now, if $map(a'_1, b'_1, \ldots, a'_n, b'_n) <_Q map(a_1, b_1, \ldots, a_n, b_n)$, there exists a j, $1 \leq j \leq n$ such for $1 \leq i < j$, $a_i = a'_i$ and $b_i = b'_i$ and $\text{Icode}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)}) > \text{Icode}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c_{\min(j,m)}, d_{\min(j,m)})$.

Part (F) now follows from part (E), and definition of X. □

We now continue with the proof of the theorem. The aim is to construct Θ which maps $coOP\_HULL^{n,m,S}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m}$ to $L^Q_{map(a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m)}$.

Note that definition of Ψ mapping grammar sequence converging to a grammar for $L^Q_{map(a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m)}$ to a grammar sequence converging to a grammar for $coOP\_HULL^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m}$ would be trivial. We thus just define Θ.

Without loss of generality, we will be giving Θ as mapping sets to sets.

For any finite $X \subseteq N^2$, let $Prop(X, a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m)$ be true iff following two properties are satisfied.

(A) $(a_1, b_1, \ldots, a_n, b_n) \in VALID_S$, $(c_1, d_1, \ldots, c_n, d_n) \in VALID_S$, and $\sum_{1\leq i\leq n} b_i < \frac{1}{\sum_{1\leq i\leq m} d_i}$.

(B) For all j, $1 \leq j \leq n$,

for all $a'_j, c'_j \in N$, $b'_j, d'_{\min(j,m)} \in S$, such that $b'_j + \sum_{1\leq i<j} b_i < \frac{1}{d'_{\min(j,m)} + \sum_{1\leq i<\min(j,m)} d_j}$,

If $\text{Icode}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)}) > \text{Icode}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c_{\min(j,m)}, d_{\min(j,m)})$, then

$[X \not\subseteq coINT\_OP\_HULL^{j,\min(j,m)}_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j;c_1,d_1,\ldots,c_{\min(j,m)-1},d_{\min(j,m)-1},c'_{\min(j,m)},d'_{\min(j,m)}}]$.

Note that whether $X, a_1, b_1, \ldots, a_n, b_n, c_1, d_1, \ldots, c_m, d_m$, satisfy (A) and (B), for all j, $1 \leq j \leq n$, is effectively testable.

Moreover, (B) above implies that,

(D) For all $(a'_1, b'_1, \ldots, a'_n, b'_n), (c'_1, d'_1, \ldots, c'_m, d'_m) \in VALID_S$, $\sum_{1\leq i\leq n} b'_i < \frac{1}{1\leq i\leq m} d'_i$, such that $map(a'_1, b'_1, \ldots, a'_n, b'_n; c'_1, d'_1, \ldots, c'_m, d'_m) <_Q map(a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m)$, $X \not\subseteq coOP\_HULL^{n,m}_{a'_1,b'_1,\ldots,a'_n,b'_n;c'_1,d'_1,\ldots,c'_m,d'_m}$.

Thus, for finite $X \subseteq N^2$, let

$\Theta(X) = \bigcup\{L^Q_{map(a_1,b_1,\ldots,a_n,b_n)} \mid Prop(X, a_1, b_1, \ldots, a_n, b_n)\}$.

For infinite X', $\Theta(X') = \bigcup_{X\subseteq X', \text{card}(X)<\infty} \Theta(X)$.

It is easy to verify that

(1) for any $X \subseteq coOP\_HULL^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m}$,

$\Theta(X) \subseteq L^Q_{map(a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m)}$ (by (D) and the fact that for any valid I and I', $map(I) <_Q map(I')$, implies $L^Q_{map(I)} \subseteq L^Q_{map(I')}$), and

(2) For any S-valid $(a_1, b_1, \ldots, a_n, b_n)$ and $(c_1, d_1, \ldots, c_m, d_m)$, by Claim 11(F) there exists a finite set $X \subseteq coOP\_HULL^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m}$, such that $\Theta(X) \supseteq L^Q_{map(a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m)}$.

Thus, we have that $\Theta(coOP\_HULL^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m}) = L^Q_{map(a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m)}$. ∎

14 Conclusions

A new complexity "scale" has been successfully applied for evaluating complexity of learning various geometrical "figures" from texts. Many upper bounds we have obtained are surprisingly lower than the ones suggested by "intuitive" learning strategies. Another surprising result is that upper and lower bounds match for *OP_HULL* and their complements, while there is a large gap between upper and lower bounds for *SEMI_HULL*s that cannot be narrowed. One more interesting aspect of this picture is that upper bounds for *OP_HULL*s, the intersection of *SEMI_HULL*s, are much lower than for *SEMI_HULL*s themselves! In general, the picture of upper and lower bounds for *OP_HULL*s and their complements is much more uniform than for *SEMI_HULL*s and their complements: for the former, *INIT*s just must be replaced with *COINIT*s, while bounds for *SEMI_HULL*s and *coSEMI_HULL*s differ even in the number of components in Q-vectors.

There are many other interesting types of geometrical concepts whose complexity can be explored in terms of Q-classes. For example, one can evaluate complexity of learning *SEMI_HULL*s and all other figures observed in our paper dropping requirement of the first angle being adjacent to x or y-axis. Even more promising seems to be the class of finite unions of *OP_HULL*s (though proofs may become technically messy). In general, we are convinced that Q-classes (possibly using some other basic classes/strategies) are very promising tools for exploring complexity of learning "hard" languages from texts.

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References

[BF72] J. Bārzdiņš and R. Freivalds. On the prediction of general recursive functions. *Soviet Mathematics Doklady*, 13:1224–1228, 1972.

[Blu67] M. Blum. A machine-independent theory of the complexity of recursive functions. *Journal of the ACM*, 14:322–336, 1967.

[Cas88] J. Case. The power of vacillation. In D. Haussler and L. Pitt, editors, *Proceedings of the Workshop on Computational Learning Theory*, pages 133–142. Morgan Kaufmann, 1988.

[CL82] J. Case and C. Lynes. Machine inductive inference and language identification. In M. Nielsen and E. M. Schmidt, editors, *Proceedings of the 9th International Colloquium on Automata, Languages and Programming*, volume 140 of *Lecture Notes in Computer Science*, pages 107–115. Springer-Verlag, 1982.

[CS83] J. Case and C. Smith. Comparison of identification criteria for machine inductive inference. *Theoretical Computer Science*, 25:193–220, 1983.

[DS86] R. Daley and C. Smith. On the complexity of inductive inference. *Information and Control*, 69:12–40, 1986.

[Fel72] J. Feldman. Some decidability results on grammatical inference and complexity. *Information and Control*, 20:244–262, 1972.

[FKS95] R. Freivalds, E. Kinber, and C. Smith. On the intrinsic complexity of learning. *Information and Computation*, 123(1):64–71, 1995.

[Gol67] E. M. Gold. Language identification in the limit. *Information and Control*, 10:447–474, 1967.

[HU79] J. Hopcroft and J. Ullman. *Introduction to Automata Theory, Languages, and Computation.* Addison-Wesley, 1979.

[JKW99] S. Jain, E. Kinber, and R. Wiehagen. Language learning from texts: Degrees of intrinsic complexity and their characterizations. Unpublihsed, 1999.

[JS96] S. Jain and A. Sharma. The intrinsic complexity of language identification. *Journal of Computer and System Sciences*, 52:393–402, 1996.

[JS97] S. Jain and A. Sharma. The structure of intrinsic complexity of learning. *Journal of Symbolic Logic*, 62:1187–1201, 1997.

[Kin94] E. Kinber. Monotonicity versus efficiency for learning languages from texts. In S. Arikawa and K. Jantke, editors, *Algorithmic Learning Theory: Fourth International Workshop on Analogical and Inductive Inference (AII '94) and Fifth International Workshop on Algorithmic Learning Theory (ALT '94)*, volume 872 of *Lecture Notes in Artificial Intelligence*, pages 395–406. Springer-Verlag, 1994.

[KS95] E. Kinber and F. Stephan. Language learning from texts: Mind changes, limited memory and monotonicity. *Information and Computation*, 123:224–241, 1995.

[LZ93] S. Lange and T. Zeugmann. Learning recursive languages with a bounded number of mind changes. *International Journal of Foundations of Computer Science*, 4(2):157–178, 1993.

[MY78] M. Machtey and P. Young. *An Introduction to the General Theory of Algorithms.* North Holland, New York, 1978.

[OW82] D. Osherson and S. Weinstein. Criteria of language learning. *Information and Control*, 52:123–138, 1982.

[Rog67] H. Rogers. *Theory of Recursive Functions and Effective Computability.* McGraw-Hill, 1967. Reprinted, MIT Press 1987.

[Wie86] R. Wiehagen. On the complexity of program synthesis from examples. *Journal of Information Processing and Cybernetics (EIK)*, 22:305–323, 1986.