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The Discrete Time Behaviour of Restricted Linear Hybrid Automata

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We summarize results from [2, 3, 1] on the discrete time behaviour of a class of restricted linear hybrid automata. Specifically, we show the regularity of the discrete time behaviour of hybrid automata in which the rates of continuous variables are governed by linear operators in a diagonal form and in which the values of the continuous variables can be observed only with finite precision. Crucially, we do not demand—as is usually done—that the values of the continuous variables be reset during mode changes. We can cope with polynomial guards and we can tolerate bounded delays both in sampling the values of the continuous variables and in effecting changes in their rates required by mode switchings. We also show that if the rates are governed by diagonalizable linear operators with rational eigenvalues *and* there is no delay in effecting rate changes, the discrete time behaviour of the hybrid automaton is recursive. However, the control state reachability problem in this setting is undecidable.

1.1. Introduction

Hybrid automata are models of control systems which consist of digital components interacting with continuously evolving physical attributes. Such control systems appear in domains such as avionics, automotive electronics and industrial process control. Hybrid automata can be used to provide effective methods for modeling and verifying the correct functioning of such systems.

A hybrid automaton is a finite transition system where each state (termed mode

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or control state) is labelled with a differential equation specifying the evolution rates of a fixed set of real-valued variables x_1, x_2, \dots, x_n . Each transition is labelled with a constraint (called a guard) on the values of x_1, x_2, \dots, x_n . A transition can be taken only when its guard is satisfied. The semantics of a hybrid automaton is given by an infinite state transition system where each state consists of the current mode and the current values of x_1, x_2, \dots, x_n . There are two types of transitions, one signifying the passage of time during which the continuous variables evolve at a mode and the other—usually instantaneous—consisting of a change of modes.

We study the behaviour of hybrid automata in which the rate functions associated with the modes are restricted linear differential equations. We show that if the values of the continuous variables can be observed only with finite precision, then the discrete time behaviour of a large class of hybrid automata is regular. Further, these behaviours can be effectively computed. The key feature of our setting is that we do *not* demand that the value of a continuous variable be reset during a mode switch. The point is, this reset assumption—often a crippling one from the modeling point of view—is often made to obtain tractable classes of hybrid automata [4, 7]. Our results suggest that focusing on discrete time semantics and the realistic assumption of finite precision leads to effective analysis methods for hybrid automata.

In the related literature, one often assumes that the rates are piecewise constant. This is so especially in settings where one does not make the reset assumption and yet obtains positive verification results [8, 11, 14]. Even here, since the mode changes can take place over continuous time (a transition may be taken any time its guard is satisfied), basic verification problems often become undecidable [4, 10]. In contrast, it was shown in [9] that one can go much further in the positive direction for piecewise constant rate automata, if one defines their behaviour using a discrete time semantics. As argued in [9], if the hybrid automaton models the closed loop system consisting of a digital controller interacting with a continuous plant, then the discrete time semantics is the natural one; the controller will observe via sensors, the states of the plant and effect, via actuators, changes in the plant dynamics at discrete time points determined by its internal clock. In [2] it was shown that, in this setting, one can in fact tolerate bounded delays both in the observation of the plant states and in effecting changes in the plant dynamics.

Both in [9] and [2], the transition guards were required to be rectangular; conjunctions of simple linear inequalities involving just one variable. In [3], it was shown that one can cope with much more expressive guards—essentially all effectively computable guards—if one further assumes that the values of the continuous variables can be observed only with finite precision. In many settings including the one where the hybrid automaton models a digital controller interacting with a continuous plant, finite precision is a natural assumption.

In [1] we showed that the combination of discrete time semantics and finite precision can not only allow more expressive guards but can also take us beyond

piecewise constant rates. One of our main results here is that under finite precision, the discrete time behaviour of a hybrid automaton is regular and effectively computable even when the rate of a continuous variable x_i in each control state q is governed by an equation of the form $dx_i/dt = c_q \cdot x_i(t)$. Further, we can cope with arbitrary computable guards. We can also tolerate bounded delays in sampling the values of the continuous variables and in effecting changes in their rates required by mode switchings. Again, it is a crucial feature that we do not demand resetting of the values of the continuous variables during mode changes.

We also showed that the discrete time behaviours of hybrid automata in a much richer setting are *recursive*. That is, for such a hybrid automaton \mathcal{A} , one can effectively determine whether a given finite sequence of control states is a discrete time behaviour of \mathcal{A} . Specifically, the rates of continuous variables at the control state q are governed by a linear differential operator represented by a diagonalizable ([12]) matrix A_q with rational eigenvalues. Further, we allow polynomial guards but do not permit delays in effecting rates changes. A consequence of this positive result is that one can effectively solve a variety of bounded model checking problems [6] in this rich setting. However, it turns out that the control state reachability problem is undecidable for this class of automata; this is so, even if the guards are restricted to be rectangular.

The proofs of the above two results seem to suggest that one can hope to go much further if update delays are allowed. This will prevent the hybrid automaton from retaining an unbounded amount of information as its dynamics evolves. However, we do not know at present how to take advantage of this observation since we lack suitable techniques for tracking rational approximations of exponential terms with *real* exponents. In this connection, the fundamental theory presented in [5] may turn to be important. We also feel that the techniques presented in [15, 16] will turn out to be useful even though they are developed under a regime where continuous variables are reset during mode changes.

In the next two sections, we define our hybrid automata and develop their discrete time semantics. Section 1.4 provides a brief summary of the results on these hybrid automata in [2, 3, 1]. In Section 1.5, we present the results in the setting of finite precision. Namely, we show that the discrete time behaviour of a class of hybrid automata is regular where the evolution rates are governed by linear operators in diagonal form. We also show that the discrete time behaviour of a class of hybrid automata is recursive where the rates matrices are diagonalizable with rational eigenvalues. In Section 1.6, we discuss the results under the assumption of perfect precision. We show that the results in 1.5 also hold in the perfect-precision setting provided that every transition guard is rectangular. We conclude in Section 1.7.

1.2. Preliminaries

Through the rest of this chapter, we fix a positive integer n and one function symbol x_i for each i in $\{1, 2, \dots, n\}$. We will often refer to the x_i 's as “continuous” variables and will view each x_i as a function (of time) $x_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$. As usual, \mathbb{R} is the set of reals and $\mathbb{R}_{\geq 0}$, the set of non-negative reals. We let \mathbb{Q} denote the set of rationals.

The transitions of the hybrid automaton will have associated guards. A *polynomial constraint* is an inequality of the form $p(x_1, x_2, \dots, x_n) \leq 0$ or $p(x_1, x_2, \dots, x_n) < 0$ where $p(x_1, x_2, \dots, x_n)$ is a polynomial over x_1, x_2, \dots, x_n with integer coefficients. A *polynomial guard* is a finite conjunction of polynomial constraints. We let Grd denote the set of polynomial guards. Unless otherwise stated, by a guard we will mean a polynomial guard.

A *valuation* V is just a member of \mathbb{R}^n . It prescribes the value $V(i)$ to the variable x_i . The notion of a valuation satisfying a guard is defined in the obvious way.

A *lazy finite-precision linear hybrid automaton* is a structure $\mathcal{A} = (Q, q_{in}, V_{in}, Delay, \epsilon, \{\rho_q\}_{q \in Q}, \{\gamma_{min}, \gamma_{max}\}, \longrightarrow)$ where:

- Q is a finite set of *control states* with q, q' ranging over Q .
- $q_{in} \in Q$ is the initial control state.
- $V_{in} \in \mathbb{Q}^n$ is the initial valuation.
- $Delay = \{\delta_{ob}^0, \delta_{ob}^1, \delta_{up}^0, \delta_{up}^1\} \subseteq \mathbb{Q}$ is the set of *delay parameters* such that $0 \leq \delta_{up}^0 \leq \delta_{up}^1 < \delta_{ob}^0 \leq \delta_{ob}^1 \leq 1$.
- ϵ , a positive rational, is the *precision of measurement*.
- $\{\rho_q\}_{q \in Q}$ is a family of rate functions associated with the control states. In the general case, ρ_q will be of the form $\dot{x} = A_q x + b_q$ where A_q is an $n \times n$ matrix with rational entries and $b_q \in \mathbb{Q}^n$. For each i in $\{1, 2, \dots, n\}$ this specifies the rate function of x_i as the differential equation $dx_i/dt = \sum_{j=1}^n A_q(i, j) \cdot x_j(t) + b_q(i)$ where $A_q(i, j)$ is the (i, j) -th entry of A_q .
- $\gamma_{min}, \gamma_{max} \in \mathbb{Q}$ are *range parameters* such that $0 < \gamma_{min} < \gamma_{max}$.
- $\longrightarrow \subseteq Q \times Grd \times Q$ is a transition relation such that $q \neq q'$ for every (q, g, q') in \longrightarrow .

Figure 1.1 displays an example of such a hybrid automaton. The control states $q1, q2$ are represented by two circles. The rate functions of $q1, q2$ are inscribed in their respective circles. The initial state $q1$ is indicated with a pointing arrow.

In what follows, for convenience we will often say “finite-precision hybrid automaton” instead of “lazy finite-precision linear hybrid automaton”. We are interested in the discrete time behaviour of our hybrid automata assuming that a suitable granularity of time has been fixed. At each time instant T_k , the automaton receives a measurement regarding the current values of the x_i 's. However, the value of x_i that is observed at time T_k is the value that held at some time $t \in [T_{k-1} + \delta_{ob}^0, T_{k-1} + \delta_{ob}^1]$. Further, the value is observed with a precision of ϵ .

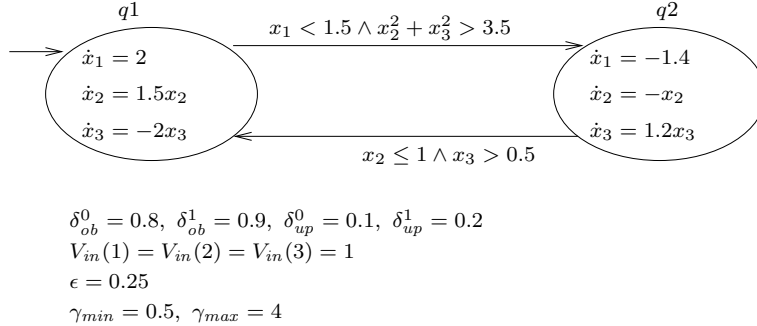


Fig. 1.1.

More specifically, any value of x_i in the half-open interval $[(m - 1/2)\epsilon, (m + 1/2)\epsilon)$ is reported as $m\epsilon$ where m is an integer. For a real number v , we will denote this rounded-off value relative to ϵ as $\langle v \rangle_\epsilon$ and often just write $\langle v \rangle$. More sophisticated rounding-off functions can also be considered (see [3]) but for ease of presentation, we shall not do so here.

If at T_k , the automaton is in control state q and the observed n -tuple of values $(\langle v_1 \rangle, \langle v_2 \rangle, \dots, \langle v_n \rangle)$ satisfies the guard g with (q, g, q') being a transition, then the automaton may perform this transition instantaneously and move to the control state q' . As a result, the x_i 's will cease to evolve according to the rate function ρ_q and instead start evolving according to the rate function $\rho_{q'}$. However, for each x_i , this change in the rate of evolution of each x_i will not kick in at T_k but at some time $t \in [T_k + \delta_{up}^0, T_k + \delta_{up}^1]$. In this sense, both the sensing of the x_i 's and the rate changes associated with mode switching take place in a lazy fashion but with bounded delays. We expect $\delta_{ob}^0, \delta_{ob}^1$ to be close to 1 and $\delta_{up}^0, \delta_{up}^1$ to be close to 0 while both $\delta_{ob}^1 - \delta_{ob}^0$ and $\delta_{up}^1 - \delta_{up}^0$ to be small compared to 1. The restriction we have imposed on the delay parameters is mainly for minimizing the notational overhead. More relaxed conditions can be imposed without affecting the main results.

In the idealized setting, the value observed at T_k is the value that holds at exactly T_k ($\delta_{ob}^0 = 1 = \delta_{ob}^1$) and the change in rates due to mode switching would kick in immediately ($\delta_{up}^0 = 0 = \delta_{up}^1$).

The parameters $\gamma_{min}, \gamma_{max}$ specify the relevant range of the absolute values of the continuous variables. The automaton gets stuck if $|x_i|$ gets outside the allowed range $[\gamma_{min}, \gamma_{max}]$ for any i . Loosely speaking, the γ_{max} bound is used to restrict the amount of information carried by a continuous variable evolving at a (positive or negative) constant rate ($\dot{x}_i = c$) and a continuous variable increasing at an exponential rate ($\dot{x}_i = c \cdot x_i(t), c > 0$). On the other hand, γ_{min} is used to restrict the amount of information carried by a continuous variable decreasing at an exponential rate ($\dot{x}_i = c \cdot x_i(t), c < 0$). We note that our setting is quite different from the classical continuous setting. Hence the standard control objective

of driving a system variable to 0 is not relevant here and thus does not pose a serious limitation.

We define a lazy *perfect-precision* differential hybrid automaton in the obvious way; the parameter ϵ is dropped and $\langle v \rangle$ is set to v for every real number v . For convenience, we shall refer to a lazy perfect-precision linear hybrid automaton as a perfect-precision hybrid automaton. For further convenience, a finite-precision hybrid automaton will often be referred to as just a hybrid automaton.

1.3. The Transition System Semantics

In what follows, we define the behaviour of one of our hybrid automata in terms of an associated transition system. The behaviour of a perfect-precision hybrid automaton will be an easy adaptation of this notion.

Through the rest of this section, we fix a hybrid automaton \mathcal{A} and assume its associated notations and terminology as defined in the previous section. A *configuration* is a triple (q, V, q') where q, q' are control states and V is a valuation. q is the current control state, q' is the control state that held at the previous time instant and V captures the *actual* values of the variables at the current time instant. The valuation V is said to be *feasible* if $\gamma_{min} \leq |V(i)| \leq \gamma_{max}$ for every i in $\{1, 2, \dots, n\}$. The configuration (q, V, q') is *feasible* iff V is a feasible valuation. The initial configuration is (q_{in}, V_{in}, q_{in}) and is assumed to be feasible. We let $Conf_{\mathcal{A}}$ denote the set of configurations. We assume that the unit of time has been fixed at some suitable level of granularity and that the rate functions $\{\rho_q\}_{q \in Q}$ have been scaled accordingly.

Suppose the automaton \mathcal{A} is in the configuration (q_k, V_k, q'_k) at time T_k . Then one unit of time will be allowed to pass and at time instant T_{k+1} , the automaton \mathcal{A} will make an instantaneous move by executing a transition or the silent action τ and move to a configuration $(q_{k+1}, V_{k+1}, q'_{k+1})$. The silent action τ will be used to record that no mode change has taken place during this move. The action μ will be used to record that a transition has been taken and as a result, a mode change has taken place. As is common, we will collapse the unit-time-passage followed by an instantaneous transition into one “time-abstract” transition labelled by τ or μ . We wish to formalize the transition relation $\Longrightarrow \subseteq Conf_{\mathcal{A}} \times \{\tau, \mu\} \times Conf_{\mathcal{A}}$. For doing so, we note that given a matrix $A \in \mathbb{Q}^{n \times n}$, a vector $b \in \mathbb{Q}^n$, a positive real T and a valuation V , we can find a unique family of curves (see [12]) $\{x_i\}_{1 \leq i \leq n}$ with $x_i : [0, T] \rightarrow \mathbb{R}$ such that for every i we have $x_i(0) = V(i)$ and for every $t \in [0, T]$ we have $dx_i/dt = \sum_{j=1}^n A_q(i, j) \cdot x_j(t) + b_q(i)$. In what follows, we shall denote the valuation $(x_1(T), x_2(T), \dots, x_n(T))$ thus obtained as $Val(A, b, T, V)$ without explicitly displaying the curves x_i 's.

Let $(q, V, q'), (q1, V1, q1')$ be in $Conf_{\mathcal{A}}$. Suppose there exist reals t_i^{up} , $i = 1, 2, \dots, n$, in $[\delta_{up}^0, \delta_{up}^1]$ such that $V1$ is related to V as follows: Let $t_{\pi_1}^{up} \leq t_{\pi_2}^{up} \leq \dots \leq t_{\pi_n}^{up}$ with $\pi_1, \pi_2, \dots, \pi_n$ being a permutation of the indices $1, 2, \dots, n$. Then

there exist valuations U_i , $i = 1, 2, \dots, n$, such that $U_1 = \text{Val}(A_{q'}, b_{q'}, t_{\pi_1}^{up}, V)$; $U_{i+1} = \text{Val}(A_i, b_i, t_{\pi_{i+1}}^{up} - t_{\pi_i}^{up}, U_i)$ for $i = 1, 2, \dots, n-1$; and $V1 = \text{Val}(A_n, b_n, 1 - t_{\pi_n}^{up}, U_n)$, where for $i = 1, 2, \dots, n$, the matrix $A_i \in \mathbb{Q}^{n \times n}$ and the vector $b_i \in \mathbb{Q}^n$ are given by: if $j \in \{\pi_1, \pi_2, \dots, \pi_i\}$, then the j -th row of A_i (b_i) equals the j -th row of A_q (b_q); otherwise the j -th row of A_i (b_i) equals the j -th row of $A_{q'}$ ($b_{q'}$).

The intuition is that at time T_{k+1} the continuous variables have valuation $V1$ while at time T_k , the continuous variables have valuation V and \mathcal{A} resides at control state q . Further, at time T_{k-1} , the automaton was at control state q' . For each i , the real number $T_k + t_i^{up}$ is the time at which x_i ceases to evolve at the rate $dx_i/dt = \sum_{j=1}^n A_{q'}(i, j) \cdot x_j + b_{q'}(i)$ and starts to evolve at the rate $dx_i/dt = \sum_{j=1}^n A_q(i, j) \cdot x_j + b_q(i)$.

Now we state the condition that \implies must fulfil. Let (q, V, q') , $(q1, V1, q1')$ be in $\text{Conf}_{\mathcal{A}}$. Suppose there exist reals t_i^{up} , $i = 1, 2, \dots, n$, in $[\delta_{up}^0, \delta_{up}^1]$ such that $V1$ is related to V as dictated above.

- (i): Suppose $q1 = q1' = q$. Then $(q, V, q') \xrightarrow{\tau} (q1, V1, q1')$.
- (ii): Suppose $q1' = q$ and there exists a transition $(q, g, q1)$ in \longrightarrow and reals t_i^{ob} , $i = 1, 2, \dots, n$, in $[\delta_{ob}^0, \delta_{ob}^1]$ such that $(\langle w_1 \rangle, \langle w_2 \rangle, \dots, \langle w_n \rangle)$ satisfies g , where w_i is the i -th component of the valuation $\text{Val}(A_n, b_n, t_i^{ob} - t_{\pi_n}^{up}, U_n)$ for $i = 1, 2, \dots, n$. Then $(q, V, q') \xrightarrow{\mu}_{\mathcal{A}} (q1, V1, q1')$.

As might be expected, the real $T_k + t_i^{ob}$ is the time at which the value of x_i was observed for each $i = 1, 2, \dots, n$. We illustrate the transitions satisfying condition (i),(ii) in figure 1.2(i) and figure 1.2(ii), respectively. The time point Δ is $T_k + \delta_{up}^0$, the shaded triangle represents $T_k + \delta_{up}^1$, \square is $T_k + \delta_{ob}^0$, and the shaded box corresponds to $T_k + \delta_{ob}^1$. For each $i = 1, 2, \dots, n$, Δ_i is $T_k + t_{\pi_i}^{up}$. The time points $\square_1, \square_2, \dots, \square_n$ form a permutation of $T_k + t_1^{ob}, T_k + t_2^{ob}, \dots, T_k + t_n^{ob}$.

Basically, there are four possible transition types depending on whether $q = q'$ and whether τ or μ is the action label. For convenience, we have collapsed these four possibilities into two cases according to τ or μ being the action label, and in each case have handled the subcases $q = q'$ and $q \neq q'$ simultaneously.

Now define the transition system $TS_{\mathcal{A}} = (RC_{\mathcal{A}}, (q_{in}, V_{in}, q_{in}), \{\tau, \mu\}, \implies_{\mathcal{A}})$ via:

- $RC_{\mathcal{A}}$, the set of *reachable configurations* of \mathcal{A} is the least subset of $\text{Conf}_{\mathcal{A}}$ that contains the initial configuration (q_{in}, V_{in}, q_{in}) and satisfies: Suppose (q, V, q') is in $RC_{\mathcal{A}}$ and is a feasible configuration. Suppose further, $(q, V, q') \xrightarrow{\alpha} (q1, V1, q1')$ for some $\alpha \in \{\tau, \mu\}$. Then $(q1, V1, q1') \in RC_{\mathcal{A}}$.
- $\implies_{\mathcal{A}}$ is \implies restricted to $RC_{\mathcal{A}} \times \{\tau, \mu\} \times RC_{\mathcal{A}}$.

We note that a reachable configuration can be the source of a transition in $TS_{\mathcal{A}}$ only if it is feasible. Thus infeasible reachable configurations will be deadlocked in $TS_{\mathcal{A}}$.

A *run* of $TS_{\mathcal{A}}$ is a finite sequence of the form

$$\sigma = (q_0, V_0, q'_0) \alpha_0 (q_1, V_1, q'_1) \alpha_1 (q_2, V_2, q'_2) \dots (q_\ell, V_\ell, q'_\ell)$$

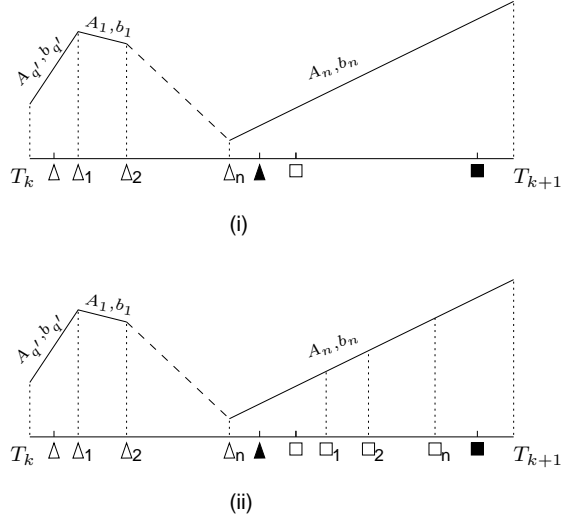


Fig. 1.2.

where (q_0, V_0, q'_0) is the initial configuration and $(q_k, V_k, q'_k) \xrightarrow{\alpha_k} \mathcal{A} (q_{k+1}, V_{k+1}, q'_{k+1})$ for $k = 0, 1, \dots, \ell - 1$. The *state sequence* induced by the run σ above is the sequence $q_0 q_1 \dots q_\ell$. We define the state sequence language of \mathcal{A} denoted $\mathcal{L}(\mathcal{A})$ to be the set of state sequences induced by runs of $TS_{\mathcal{A}}$.

Lastly, we remark that the above transition system $TS_{\mathcal{A}}$ and associated definitions apply in exactly the same way to a perfect-precision hybrid automaton, except that $\langle v \rangle$ is set to v for every real number v .

1.4. Summary of the Results

In [3], it is shown that, if each continuous variable evolves at (possibly different) constant rates in all the control states (each ρ_q takes the form $\dot{x} = b_q$), then the discrete time behaviour of a finite-precision hybrid automaton is regular and can be effectively computed. This result is generalized in [1], where we showed that for the case that each A_q is a diagonal matrix, the control state sequence language generated by a finite-precision hybrid automaton is regular and can be effectively computed. In fact this holds provided every continuous variable either evolves at constant rates in all the control states or at exponential rates in all the control states.

In [1], we also consider the more general case that each A_q is a diagonalizable matrix and show that, with the additional restriction that there are no delays associated with rates update ($\delta_{up}^0 = 0 = \delta_{up}^1$), the control state sequence language generated by a finite-precision hybrid automaton is recursive. However, the control state reachability problem in this setting (with the restriction of no update delays)

is undecidable.

The setting of perfect precision was studied in [2]. There it was shown that if every continuous variable evolves at constant rates in all the control states, then the control state sequence language generated by a perfect-precision hybrid automaton is regular and can be effectively computed, provided that every transition guard is *rectangular*. A *rectangular guard* is a finite conjunction of inequalities of the forms $x_i < c$, $x_i \leq c$, $x_i > c$, $x_i \geq c$, where $i \in \{1, 2, \dots, n\}$, and c is a rational number.

For more general rate functions, it can be shown that each of the three main results of [1] described above also holds for perfect-precision hybrid automata provided every transition guard is rectangular.

1.5. The Finite-Precision Setting

In this section, we present in a unified way the results from [3, 1] on finite-precision hybrid automata.

1.5.1. The Constant Rates Case

It was shown in [3] that if every continuous variable evolves at (possibly different) constant rates in all the control states, then the language of state sequences of a finite-precision hybrid automaton is regular. More precisely:

Theorem 1.1 ([3]). *Let \mathcal{A} be a hybrid automaton such that for each q , ρ_q takes the form $\dot{x} = b_q$ (that is, each A_q is the zero matrix). Then $\mathcal{L}(\mathcal{A})$ is a regular subset of Q^* . Further, a finite state automaton accepting $\mathcal{L}(\mathcal{A})$ can be effectively computed from a given presentation of \mathcal{A} .*

Proof: The proof consists of two major steps. The first one is to quotient the set of reachable configurations $RC_{\mathcal{A}}$ into a *finite number* of equivalence classes using a suitably chosen equivalence relation \approx . The crucial property required of \approx is that it should be a congruence with respect to the transition relation of $TS_{\mathcal{A}}$. In other words, if $(q1, V1, q1') \approx (q2, V2, q2')$ and $(q1, V1, q1') \xrightarrow{\alpha}_{\mathcal{A}} (q3, V3, q3')$, then we require that there exists a configuration $(q4, V4, q4')$ such that $(q2, V2, q2') \xrightarrow{\alpha}_{\mathcal{A}} (q4, V4, q4')$ and $(q3, V3, q3') \approx (q4, V4, q4')$. The second step is to show that we can effectively compute these equivalence classes and a transition relation over them such that the resulting finite state automaton generates the language of control state sequences.

We assume without loss of generality that for each guard g in \mathcal{A} , the valuation V satisfies g only if V is feasible.

To define the equivalence relation \approx , we shall partition the real line into a *finite* number of intervals and assert that $(q1, V1, q1') \approx (q2, V2, q2')$ iff $q1 = q2$ and $V1(i)$ falls into the same interval as $V2(i)$ for every i , and $q1' = q2'$. More precisely, we shall firstly construct a *finite* set Θ of rational numbers. Secondly, list the members

of Θ in *increasing* order as $\theta_1, \theta_2, \dots, \theta_m$ where $m = |\Theta|$, and divide the real line into finitely many open intervals $\{(-\infty, \theta_1), (\theta_1, \theta_2), \dots, (\theta_{m-1}, \theta_m), (\theta_m, \infty)\}$, and finitely many singleton intervals $[\theta_j, \theta_j], j = 1, 2, \dots, m$. Let \mathcal{I} denote the collection of these open and singleton intervals. Define the map $\|\cdot\| : \mathbb{R} \rightarrow \mathcal{I}$ via: $\|v\| = I$ iff $v \in I$. We extend the map $\|\cdot\|$ to valuations in the obvious way, and denote this extension also as $\|\cdot\|$. Namely, $\|V\| = (\|V(1)\|, \|V(2)\|, \dots, \|V(n)\|)$ for a valuation V . It follows that $(q1, V1, q1') \approx (q2, V2, q2')$ iff $q1 = q2, \|V1\| = \|V2\|, q1' = q2'$.

We now describe Θ . Let Δ to be the largest positive rational number that *integrally* divides every number in the set of rational numbers $\{\delta_{ob}^0, \delta_{ob}^1, \delta_{up}^0, \delta_{up}^1, 1\}$. Define Γ to be the largest rational which *integrally* divides every number in the finite set of rational numbers $\{b_q(i) \cdot \Delta \mid q \in Q, i \in \{1, 2, \dots, n\}\} \cup \{\gamma_{min}, \gamma_{max}\} \cup \{\epsilon/2\}$. Let \mathbb{Z} denote the set of integers. We define Θ to be the *finite* set of rational numbers $\{h\Gamma \in [-\gamma_{max}, \gamma_{max}] \mid h \in \mathbb{Z}\}$. In other words, Θ contains integral multiples of Γ in the interval $[-\gamma_{max}, \gamma_{max}]$.

Next we show that the equivalence relation \approx defined above is indeed a congruence relation.

Claim 1.1. *Suppose $(q1, V1, q1') \approx (q2, V2, q2')$ and $(q1, V1, q1') \xrightarrow{\alpha}_{\mathcal{A}} (q3, V3, q3')$, then there exists a reachable configuration $(q4, V4, q4')$ such that $(q2, V2, q2') \xrightarrow{\alpha}_{\mathcal{A}} (q4, V4, q4')$ and $(q3, V3, q3') \approx (q4, V4, q4')$.*

Proof of Claim 1.1: Clearly $q1 = q2$ and $q1' = q2'$. Set $q4 = q3$ and $q4' = q3'$. We show that $(q2, V2, q2')$ is a feasible configuration and there exists a valuation $V4$ such that $(q2, V2, q2') \xrightarrow{\alpha}_{\mathcal{A}} (q4, V4, q4')$ and $\|V4\| = \|V3\|$.

We first note that the configuration $(q2, V2, q2')$ is feasible. Since the configuration $(q1, V1, q1')$ is feasible, we have $\gamma_{min} \leq |V1(i)| \leq \gamma_{max}$ for each i . Since $-\gamma_{min}, \gamma_{min}, -\gamma_{max}, \gamma_{max}$ are members of Θ and $\|V1\| = \|V2\|$, we conclude that $\gamma_{min} \leq |V2(i)| \leq \gamma_{max}$ for each i .

We show the existence of $V4$ by considering two cases according to $\alpha = \tau$ or $\alpha = \mu$.

—**Case 1:** $\alpha = \tau$.

It follows from the definition of $TS_{\mathcal{A}}$ that there exist reals $t_i^{up} \in [\delta_{up}^0, \delta_{up}^1]$, $i = 1, 2, \dots, n$, such that $V3(i) = V1(i) + b_{q1'}(i) \cdot t_i^{up} + b_{q1}(i) \cdot (1 - t_i^{up})$ for every i . It suffices to show that there exist reals $\hat{t}_i^{up} \in [\delta_{up}^0, \delta_{up}^1]$, $i = 1, 2, \dots, n$, such that $\|V4(i)\| = \|V3(i)\|$, where $V4(i) = V2(i) + b_{q1'}(i) \cdot \hat{t}_i^{up} + b_{q1}(i) \cdot (1 - \hat{t}_i^{up})$.

We fix an $i \in \{1, 2, \dots, n\}$ and show the existence of \hat{t}_i^{up} . The argument applies to any i . Assume $\|V3(i)\|_{dif} = (\theta, \theta')$ where $\theta, \theta' \in \Theta$ and $b_{q1'}(i) > b_{q1}(i)$. It will become clear that other cases can be similarly handled. For any real v , let $\Phi^\tau(v)$ be the condition

$$\begin{aligned} \exists t^{up} \in \mathbb{R}. \quad & \delta_{up}^0 \leq t^{up} \leq \delta_{up}^1 \\ & \wedge \theta < v + b_{q1'}(i) \cdot t^{up} + b_{q1}(i) \cdot (1 - t^{up}) < \theta' . \end{aligned}$$

It is easy to see that $\Phi^\tau(v)$ holds iff $\eta < v < \eta'$ where $\eta = \theta - b_{q1'}(i) \cdot \delta_{up}^1 - b_{q1}(i) \cdot (1 - \delta_{up}^1)$ and $\eta' = \theta' - b_{q1'}(i) \cdot \delta_{up}^0 - b_{q1}(i) \cdot (1 - \delta_{up}^0)$.

Since $\Phi^\tau(V1(i))$ holds, we have $\eta < V1(i) < \eta'$. Note that η, η' are members of Θ (if $\eta, \eta' \in [-\gamma_{max}, \gamma_{max}]$). Applying $\|V2(i)\| = \|V1(i)\|$ then yields $\eta < V2(i) < \eta'$ and consequently $\Phi^\tau(V2(i))$ holds. This establishes the existence of \hat{t}_i^{up} for any $i \in \{1, 2, \dots, n\}$.

—**Case 2:** $\alpha = \mu$.

Suppose that in the transition $(q1, V1, q1') \xRightarrow{\mu}_{\mathcal{A}} (q3, V3, q3')$, for each i , the rate update of x_i (from $b_{q1'}(i)$ to $b_{q1}(i)$) kicks in at time $T_k + t_i^{up}$ and the value of x_i was observed at time $T_k + t_i^{ob}$, where T_k is the time at which \mathcal{A} assumes the configuration $(q1, V1, q1')$. Let w_i be the (actual) value of x_i at time $T_k + t_i^{ob}$ for each i and $(q1, g, q3)$ be a transition of \mathcal{A} where $(\langle w_1 \rangle, \dots, \langle w_n \rangle)$ satisfies g . We show that there exist reals \hat{t}_i^{up} in $[\delta_{up}^0, \delta_{up}^1]$, \hat{t}_i^{ob} in $[\delta_{ob}^0, \delta_{ob}^1]$, $i = 1, 2, \dots, n$, which fulfil the condition detailed as follows. Suppose \mathcal{A} assumes configuration $(q2, V2, q2')$ at time \hat{T}_k , and the rate update of x_i (from $b_{q2'}(i)$ to $b_{q2}(i)$) kicks in at time $\hat{T}_k + \hat{t}_i^{up}$, and the value of x_i is observed at time $\hat{T}_k + \hat{t}_i^{ob}$. Let \hat{w}_i be the actual value of x_i at time $\hat{T}_k + \hat{t}_i^{ob}$ for each i . Let $V4$ be the valuation of x_1, x_2, \dots, x_n at time \hat{T}_{k+1} . Then for each i , $\|\hat{w}_i\| = \|w_i\|$ And $\|V4(i)\| = \|V3(i)\|$.

We then have $\langle \hat{w}_i \rangle = \langle w_i \rangle$ for each i , following the fact that $(m + 1/2)\epsilon \in \Theta$ for integers m with $\gamma_{min} \leq \|(m + 1/2)\epsilon\| \leq \gamma_{max}$. Thus $(\langle \hat{w}_1 \rangle, \langle \hat{w}_2 \rangle, \dots, \langle \hat{w}_n \rangle)$ also satisfies the guard g . One can then conclude that the configuration $(q4, V4, q4')$ with the chosen $V4$ possesses the desired property.

We now fix an $i \in \{1, 2, \dots, n\}$ and show the existence of $\hat{t}_i^{up}, \hat{t}_i^{ob}$. The argument applies to any i . Assume that $b_{q1'}(i) > b_{q1}(i) > 0$, $\|V3(i)\| = (\theta, \theta')$ and $\|w_i\| = (\vartheta, \vartheta')$, where $\theta, \theta', \vartheta, \vartheta' \in \Theta$. Other cases can be similarly handled.

It follows from the definition of $TS_{\mathcal{A}}$ that $\Phi^\mu(V1(i))$ holds, where $\Phi^\mu(v)$ is the condition

$$\begin{aligned} \exists t^{up} \in \mathbb{R}. \exists t^{ob} \in \mathbb{R}. \quad & \delta_{up}^0 \leq t^{up} \leq \delta_{up}^1 \\ & \bigwedge \theta < v + b_{q1'}(i) \cdot t^{up} + b_{q1}(i) \cdot (1 - t^{up}) < \theta' \\ & \bigwedge \delta_{ob}^0 \leq t^{ob} \leq \delta_{ob}^1 \\ & \bigwedge \vartheta < v + b_{q1'}(i) \cdot t^{up} + b_{q1}(i) \cdot (t^{ob} - t^{up}) < \vartheta' . \end{aligned}$$

It is not difficult to see that the condition $\Phi^\mu(v)$ is in fact equivalent to $\eta < v < \eta'$, where η is the larger of $\theta - b_{q1'}(i) \cdot \delta_{up}^1 - b_{q1}(i) \cdot (1 - \delta_{up}^1)$ and $\vartheta - b_{q1'}(i) \cdot \delta_{up}^1 - b_{q1}(i) \cdot (\delta_{ob}^1 - \delta_{up}^1)$. On the other hand, η' is the smaller of $\theta' - b_{q1'}(i) \cdot \delta_{up}^0 - b_{q1}(i) \cdot (1 - \delta_{up}^0)$ and $\vartheta' - b_{q1'}(i) \cdot \delta_{up}^0 - b_{q1}(i) \cdot (\delta_{ob}^0 - \delta_{up}^0)$. Thus, both η, η' are members of Θ (if $\eta, \eta' \in [-\gamma_{max}, \gamma_{max}]$). Consequently, $\Phi^\mu(V2(i))$ also holds and this establishes the existence of reals $\hat{t}_i^{up}, \hat{t}_i^{ob}$. \square

Having established the claim that \approx is a congruence with respect to $\xRightarrow{\mu}_{\mathcal{A}}$, we now argue that one can effectively construct a finite automaton \mathcal{B} which accepts $\mathcal{L}(\mathcal{A})$. Clearly the finitely many equivalence classes of \approx can be effectively represented. Now construct a finite transition system \mathcal{B} whose states are the finitely many equivalence classes of \approx . Further, there is a transition from $C1$ to $C2$ with label α iff

there exists (q, V, q') in $C1$, $(q1, V1, q1')$ in $C2$ such that $(q, V, q') \xrightarrow{\alpha}_{\mathcal{A}} (q1, V1, q1')$. We note that the transition relation of \mathcal{B} can be effectively computed. Let $C1, C2$ be states of \mathcal{B} . Suppose $C1$ consists of those configurations $(q1, V1, q1')$ where $q1, q1' \in Q$ and $\|V1(i)\| = (\theta_i, \theta'_i)$ with $\theta_i, \theta'_i \in \Theta$ for each $i = 1, 2, \dots, n$. Let $\alpha = \tau$. Other cases can be similarly handled. To determine whether there exists a transition from $C1$ to $C2$ with label τ , we pick the configuration $(q1, U1, q1')$ from $C1$ with $U1(i) = \frac{1}{2}(\theta_i + \theta'_i)$ for each i , and check whether there exists a valuation $U2$ such that $\|U2\| = \|V2\|$ and moreover $(q1, U1, q1') \xrightarrow{\tau}_{\mathcal{A}} (q2, U2, q2')$. As in the proof of claim 1.1, the existence of $U2$ can be effectively determined by utilizing the condition $\Phi^\tau(v)$ described therein.

One can then straightforwardly construct from \mathcal{B} a finite state automaton which accepts the language $\mathcal{L}(\mathcal{A})$. This completes the proof of theorem 1.1. \square

We wish to point out that in [3], theorem 1.1 was shown for the slightly more general case where γ_{min} is allowed to be zero. As remarked earlier, the bound γ_{min} is used to bound the amount of information carried by a continuous variable decreasing at an exponential rate. Thus, the bound γ_{min} does not play a role in theorem 1.1.

1.5.2. The Exponential Rates Case

In [1], we generalize theorem 1.1 to the case that each A_q is a diagonal matrix but where every continuous variable either evolves at constant rates in all the modes or at exponential rates in all the modes. More precisely, we have:

Theorem 1.2 ([1]). *Let \mathcal{A} be a finite-precision hybrid automaton such that A_q is a diagonal matrix for every control state q . Suppose there exists a fixed partition $\{DIF, CON\}$ of the indices $\{1, 2, \dots, n\}$ such that for each control state q , ρ_q is such that $\dot{x}_i = A_q(i, i) \cdot x_i$ if $i \in DIF$ and $\dot{x}_i = b_q(i)$ if $i \in CON$. Then $\mathcal{L}(\mathcal{A})$ is a regular subset of Q^* . Further, a finite state automaton accepting $\mathcal{L}(\mathcal{A})$ can be effectively computed from \mathcal{A} .*

Proof: The basic strategy is to extend the proof of theorem 1.1, to deal with continuous variables evolving at exponential rates. The simple but key observation that enables this is, in the (natural) logarithmic scale, exponential rates get represented as *constant* rates. However, this logarithmic representation brings a new complication. As in the proof of theorem 1.5, due to the guards and the bounds $\gamma_{min}, \gamma_{max}$, one need to handle inequalities involving comparisons of x_i with $\gamma_{min}, \gamma_{max}, (m + 1/2)\epsilon$, where m is an integer. Thus, to partition the real line into finitely many intervals, one need to take into account *irrational* numbers of the forms $\ln \gamma_{min}, \ln \gamma_{max}, \ln((m + 1/2)\epsilon)$.

More precisely, we shall define a finite set Θ_{con} of rational numbers relevant to variables x_i with $i \in CON$, and a finite set Θ_{dif} of real numbers relevant to

variables x_i with $i \in DIF$. As in the proof of theorem 1.5, we list members of Θ_{con} in increasing order and divide the real line into finitely many open intervals and finitely many singleton intervals. Let \mathcal{I}_{con} denote the collection of these intervals. Do the same for Θ_{dif} . Let \mathcal{I}_{dif} denote the collection intervals defined via Θ_{dif} .

For notational convenience, we assume $V_{in}(i) > 0$ for every $i \in DIF$. The key consequence of this assumption is that in any reachable configuration, the value of x_i for $i \in DIF$ will be positive.

Let \mathbb{R}^+ be the set of positive reals. Define the map $\|\cdot\|_{dif} : \mathbb{R}^+ \rightarrow \mathcal{I}_{dif}$ via: $\|v\| = I$ if $\ln v \in I$. Define $\|\cdot\|_{con} : \mathbb{R} \rightarrow \mathcal{I}_{con}$ via: $\|v\| = I$ if $v \in I$. Finally we define $\|\cdot\| : RC_{\mathcal{A}} \rightarrow (\mathcal{I}_{dif} \cup \mathcal{I}_{con})^n$ by: $\|V\| = (I_1, I_2, \dots, I_n)$ where $I_i = \|V(i)\|_{dif}$ for $i \in DIF$ and $I_i = \|V(i)\|_{con}$ for $i \in CON$. Now define the equivalence relation \approx over configurations via: $(q1, V1, q1') \approx (q2, V2, q2')$ iff $q1 = q2$, $\|V1\| = \|V2\|$ and $q1' = q2'$.

We describe $\Theta_{con}, \Theta_{dif}$. Pick Δ as in the proof of theorem 1.5. Define Γ to be the largest rational which *integrally* divides every number in the finite set of rational numbers $\{A_q(i, i) \cdot \Delta \mid q \in Q, i \in DIF\} \cup \{b_q(j) \cdot \Delta \mid q \in Q, j \in CON\} \cup \{\gamma_{min}, \gamma_{max}\} \cup \{\epsilon/2\}$. Define Θ_{con} to be the *finite* set of rational numbers $\{h\Gamma \in [-\gamma_{max}, \gamma_{max}] \mid h \in \mathbb{Z}\}$. Let Θ_{IR} be the set of irrational numbers $\{\ln((m+1/2)\epsilon) \mid m \in \mathbb{Z}, \langle \gamma_{min} \rangle \leq m\epsilon \leq \langle \gamma_{max} \rangle\} \cup \{\ln \gamma_{min}, \ln \gamma_{max}\}$. Define Θ_{dif} to be the *finite* set of real numbers $\{h\Gamma \in [\ln \gamma_{min}, \ln \gamma_{max}] \mid h \in \mathbb{Z}\} \cup \{\ell\Gamma + \theta \in [\ln \gamma_{min}, \ln \gamma_{max}] \mid \ell \in \mathbb{Z}, \theta \in \Theta_{IR}\}$. In other words, Θ_{dif} contains rational numbers of the form $h\Gamma$ in the interval $[\ln \gamma_{min}, \ln \gamma_{max}]$ where h is a (positive) integer, and irrational numbers of the form $\ell\Gamma + \theta$ in the interval $[\ln \gamma_{min}, \ln \gamma_{max}]$ where ℓ is an integer (that can be positive, zero or negative) and θ is a member of Θ_{IR} .

Similar to the proof of claim 1.1, one can then argue that the equivalence relation \approx defined above is a congruence relation with respect to the transition relation of $TS_{\mathcal{A}}$.

Claim 1.2. *Suppose $(q1, V1, q1') \approx (q2, V2, q2')$ and $(q1, V1, q1') \xrightarrow{\alpha}_{\mathcal{A}} (q3, V3, q3')$, then there exists a reachable configuration $(q4, V4, q4')$ such that $(q2, V2, q2') \xrightarrow{\alpha}_{\mathcal{A}} (q4, V4, q4')$ and $(q3, V3, q3') \approx (q4, V4, q4')$.*

Proof of Claim 1.2: Clearly $q1 = q2$ and $q1' = q2'$. Set $q4 = q3$ and $q4' = q3'$. We show that $(q2, V2, q2')$ is a feasible configuration and there exists a valuation $V4$ such that $(q2, V2, q2') \xrightarrow{\alpha}_{\mathcal{A}} (q4, V4, q4')$ and $\|V4\| = \|V3\|$.

We first note that the configuration $(q2, V2, q2')$ is feasible. Fix an $i \in DIF$. Since the configuration $(q1, V1, q1')$ is feasible, we have $\ln \gamma_{min} \leq \ln V1(i) \leq \ln \gamma_{max}$. Since $\ln \gamma_{min}, \ln \gamma_{max}$ are members of Θ_{dif} and $\|V1(i)\|_{dif} = \|V2(i)\|_{dif}$, we conclude $\ln \gamma_{min} \leq \ln V2(i) \leq \ln \gamma_{max}$ and so $\gamma_{min} \leq |V2(i)| \leq \gamma_{max}$. Similarly it is easy to see that $\gamma_{min} \leq |V2(i)| \leq \gamma_{max}$ for $i \in CON$.

We show the existence of $V4$ by considering two cases according to $\alpha = \tau$ or $\alpha = \mu$.

—**Case 1:** $\alpha = \tau$.

It follows from the definition of $TS_{\mathcal{A}}$ that there exist reals $t_i^{up} \in [\delta_{up}^0, \delta_{up}^1]$, $i = 1, 2, \dots, n$, such that $\ln V3(i) = \ln V1(i) + A_{q1'}(i, i) \cdot t_i^{up} + A_{q1}(i, i) \cdot (1 - t_i^{up})$ for $i \in DIF$ and $V3(i) = V1(i) + b_{q1'}(i) \cdot t_i^{up} + b_{q1}(i) \cdot (1 - t_i^{up})$ for $i \in CON$. It suffices to show that there exist reals $\hat{t}_i^{up} \in [\delta_{up}^0, \delta_{up}^1]$, $i = 1, 2, \dots, n$, such that $\|V4\| = \|V3\|$, where $\ln V4(i) = \ln V2(i) + A_{q1'}(i, i) \cdot \hat{t}_i^{up} + A_{q1}(i, i) \cdot (1 - \hat{t}_i^{up})$ for $i \in DIF$ and $V4(i) = V2(i) + b_{q1'}(i) \cdot \hat{t}_i^{up} + b_{q1}(i) \cdot (1 - \hat{t}_i^{up})$ for $i \in CON$.

In what follows, we will often need to give similar arguments for $i \in DIF$ and $i \in CON$. To avoid repetition, we will omit the latter.

Fix an $i \in DIF$. We show the existence of \hat{t}_i^{up} . Assume $\|V3(i)\|_{dif} = (\theta, \theta')$ where $\theta, \theta' \in \Theta_{dif}$ and $A_{q1'}(i, i) > A_{q1}(i, i)$. It will become clear that other cases can be similarly handled. For any real v , let $\Phi^\tau(v)$ be the condition

$$\begin{aligned} \exists t^{up} \in \mathbb{R}. \quad & \delta_{up}^0 \leq t^{up} \leq \delta_{up}^1 \\ & \wedge \theta < v + A_{q1'}(i, i) \cdot t^{up} + A_{q1}(i, i) \cdot (1 - t^{up}) < \theta' . \end{aligned}$$

It is easy to see that $\Phi^\tau(v)$ holds iff $\eta < v < \eta'$ where $\eta = \theta - A_{q1'}(i, i) \cdot \delta_{up}^1 - A_{q1}(i, i) \cdot (1 - \delta_{up}^1)$ and $\eta' = \theta' - A_{q1'}(i, i) \cdot \delta_{up}^0 - A_{q1}(i, i) \cdot (1 - \delta_{up}^0)$.

Since $\Phi^\tau(\ln V1(i))$ holds, we have $\eta < \ln V1(i) < \eta'$. Note that η, η' are members of Θ_{dif} (if $\eta, \eta' \in [\ln \gamma_{min}, \ln \gamma_{max}]$). Applying $\|V2(i)\|_{dif} = \|V1(i)\|_{dif}$ then yields $\eta < \ln V2(i) < \eta'$ and consequently $\Phi^\tau(\ln V2(i))$ holds. This establishes the existence of \hat{t}_i^{up} for $i \in DIF$.

—**Case 2:** $\alpha = \mu$.

As in Case 1, it follows from the definition of $TS_{\mathcal{A}}$ that there exist reals t_i^{up} in $[\delta_{up}^0, \delta_{up}^1]$, $i = 1, 2, \dots, n$, such that $\ln V3(i) = \ln V1(i) + A_{q1'}(i, i) \cdot t_i^{up} + A_{q1}(i, i) \cdot (1 - t_i^{up})$ for $i \in DIF$ and $V3(i) = V1(i) + b_{q1'}(i) \cdot t_i^{up} + b_{q1}(i) \cdot (1 - t_i^{up})$ for $i \in CON$. Further there exist reals $t_i^{ob} \in [\delta_{ob}^0, \delta_{ob}^1]$, $i = 1, 2, \dots, n$, and a guard g such that the following conditions are satisfied: Firstly, $(q1, g, q3) \in \longrightarrow$. Secondly, $(\langle U(1) \rangle, \langle U(2) \rangle, \dots, \langle U(n) \rangle)$ satisfies g , where U is the valuation with $\ln U(i) = \ln V1(i) + A_{q1'}(i, i) \cdot t_i^{up} + A_{q1}(i, i) \cdot (t_i^{ob} - t_i^{up})$ for $i \in DIF$; $U(i) = V1(i) + b_{q1'}(i) \cdot t_i^{up} + b_{q1}(i) \cdot (t_i^{ob} - t_i^{up})$ for $i \in CON$. We shall show the existence of reals $\hat{t}_i^{up} \in [\delta_{up}^0, \delta_{up}^1]$, $\hat{t}_i^{ob} \in [\delta_{ob}^0, \delta_{ob}^1]$, $i = 1, 2, \dots, n$, such that $\|V4\| = \|V3\|$ and $\|U'\| = \|U\|$ where $V4$ is the valuation given by $\ln V4(i) = \ln V2(i) + A_{q1'}(i, i) \cdot \hat{t}_i^{up} + A_{q1}(i, i) \cdot (1 - \hat{t}_i^{up})$ for $i \in DIF$ and $V4(i) = V2(i) + b_{q1'}(i) \cdot \hat{t}_i^{up} + b_{q1}(i) \cdot (1 - \hat{t}_i^{up})$ for $i \in CON$. And U' is the valuation given by $\ln U'(i) = \ln V2(i) + A_{q1'}(i, i) \cdot \hat{t}_i^{up} + A_{q1}(i, i) \cdot (\hat{t}_i^{ob} - \hat{t}_i^{up})$ for $i \in DIF$ and $U'(i) = V2(i) + b_{q1'}(i) \cdot \hat{t}_i^{up} + b_{q1}(i) \cdot (\hat{t}_i^{ob} - \hat{t}_i^{up})$ for $i \in CON$. First we argue that the existence of U' satisfying $\|U'\| = \|U\|$ will guarantee $\langle U'(i) \rangle = \langle U(i) \rangle$ for $i = 1, 2, \dots, n$. This follows from the fact $\ln((m + 1/2)\epsilon) \in \Theta_{dif}$ for integers m with $\langle \gamma_{min} \rangle \leq m\epsilon \leq \langle \gamma_{max} \rangle$, and $(m' + 1/2)\epsilon \in \Theta_{con}$ for integers m' with $\langle -\gamma_{max} \rangle \leq m'\epsilon \leq \langle \gamma_{max} \rangle$. Thus U' also satisfies the guard g since U satisfies g . So the existence of $\hat{t}_i^{up}, \hat{t}_i^{ob}$, $i = 1, 2, \dots, n$, suffices to establish the claim.

Fix an $i \in DIF$. Assume $\|V3(i)\|_{dif} = (\theta, \theta')$, $\|U(i)\|_{dif} = (\vartheta, \vartheta')$ where $\theta, \theta', \vartheta, \vartheta' \in \Theta_{dif}$ and $A_{q1'}(i, i) > A_{q1}(i, i) > 0$. Other cases can be similarly

handled. For any real v , let $\Phi^\mu(v)$ be the condition

$$\begin{aligned} \exists t^{up} \in \mathbb{R}. \exists t^{ob} \in \mathbb{R}. \quad & \delta_{up}^0 \leq t^{up} \leq \delta_{up}^1 \\ & \bigwedge \theta < v + A_{q1'}(i, i) \cdot t^{up} + A_{q1}(i, i) \cdot (1 - t^{up}) < \theta' \\ & \bigwedge \delta_{ob}^0 \leq t^{ob} \leq \delta_{ob}^1 \\ & \bigwedge \vartheta < v + A_{q1'}(i, i) \cdot t^{up} + A_{q1}(i, i) \cdot (t^{ob} - t^{up}) < \vartheta' . \end{aligned}$$

As in Case 1, it is easy to see that $\Phi^\mu(v)$ holds iff $\eta < u < \eta'$, where η is the larger of $\theta - A_{q1'}(i, i) \cdot \delta_{up}^1 - A_{q1}(i, i) \cdot (1 - \delta_{up}^1)$ and $\vartheta - A_{q1'}(i, i) \cdot \delta_{up}^1 - A_{q1}(i, i) \cdot (\delta_{ob}^1 - \delta_{up}^1)$. On the other hand, η' is the smaller of $\theta' - A_{q1'}(i, i) \cdot \delta_{up}^0 - A_{q1}(i, i) \cdot (1 - \delta_{up}^0)$ and $\vartheta' - A_{q1'}(i, i) \cdot \delta_{up}^0 - A_{q1}(i, i) \cdot (\delta_{ob}^0 - \delta_{up}^0)$. It follows that η, η' are members of Θ_{dif} (if $\eta, \eta' \in [\ln \gamma_{min}, \ln \gamma_{max}]$). Thus, as in Case 1, one concludes that $\Phi^\mu(\ln V2(i))$ holds and the existence of $\hat{t}_i^{up}, \hat{t}_i^{ob}$ for $i \in DIF$ is established.

By filling in similar but simpler arguments for $i \in CON$, we can complete the proof of claim 1.2. \square

Having established the claim that \approx is a congruence with respect to $\implies_{\mathcal{A}}$, we now argue that one can effectively construct a finite automaton which accepts $\mathcal{L}(\mathcal{A})$. Clearly, the members of Θ_{dif} and Θ_{con} can be effectively represented. Further, the members of $\Theta_{dif}(\Theta_{con})$ can be effectively ordered and thus the finitely many equivalence classes of \approx can be effectively represented. Note that, to compare two members of Θ_{dif} one just needs to determine whether $e^{m_1} < m_2$ for integers m_1, m_2 . This can be done by approximating e sufficiently precisely using for instance the power series expansion of e . Now construct a finite transition system \mathcal{B} whose states are the finitely many equivalence classes of \approx . Further, there is a transition from $C1$ to $C2$ with label α iff there exists (q, V, q') in $C1$, $(q1, V1, q1')$ in $C2$ such that $(q, V, q') \xrightarrow{\alpha}_{\mathcal{A}} (q1, V1, q1')$. From the proof of claim 1.2, it is not difficult to see that to determine whether there exists a transition from $C1$ to $C2$ with label α amounts to comparing members of Θ_{dif} (and Θ_{con}). Hence the transition system \mathcal{B} can be effectively computed. It is now straightforward to construct from \mathcal{B} a finite state automaton which accepts $\mathcal{L}(\mathcal{A})$. This completes the proof of theorem 1.2. \square

1.5.3. Diagonalizable Rate Matrices

Here we describe the investigation in [1] for the more general case that each A_q is diagonalizable [12] and has n distinct rational eigenvalues. Recall that the $n \times n$ matrix A is diagonalizable in case there is a basis of eigenvectors $\{f_1, f_2, \dots, f_n\}$ so that under the associated coordinate transformation, A can be represented as the diagonal matrix $diag(\lambda_1, \lambda_2, \dots, \lambda_n)$ with the λ_i 's being the eigenvalues of A . The demand on each A_q to have n distinct rational eigenvalues is natural given the concern for effective computations.

We further restrict ourselves to the case where there is no delay associated with the update of rates of the continuous variables ($\delta_{up}^0 = 0 = \delta_{up}^1$). This is due to the

fact at present we don't know how to deal with differential equations of the form $\dot{x} = Ax + b$. One will have to deal with such equations if update delays are present ($\delta_{up}^0 < \delta_{up}^1$). This is due to the fact that the rate changes of the continuous variables may kick in at *different* times in the interval $[T_k + \delta_{up}^0, T_k + \delta_{up}^1]$.

With the above restrictions in place, [1] showed that the control state sequence language generated by a finite-precision hybrid automaton is recursive. However, [1] also showed that the control state reachability problem is undecidable in this setting. We now state and explain these two results.

Theorem 1.3. *Suppose \mathcal{A} is a finite-precision hybrid automaton such that $\delta_{up}^0 = 0 = \delta_{up}^1$ and for every control state q , A_q is a diagonalizable matrix having n distinct rational eigenvalues. Then $\mathcal{L}(\mathcal{A})$ is a recursive subset of Q^* .*

Proof: First we note that the first order theory of the reals augmented with the constant e is decidable. For convenience we shall denote this augmented structure as $(\mathbb{R}, +, \cdot, <, 0, 1, e)$ but emphasize that e , the base of the natural logarithm is being used as a constant and *not* as an operator. To see that the augmented theory is decidable, we observe that one can effectively determine whether $p(e) < 0$ for any given polynomial $p(e)$ with integer coefficients. Since $e = 1 + \sum_{h=1}^{\infty} 1/h!$, we have

$$1 + \sum_{h=1}^k \frac{1}{h!} < e < 1 + \sum_{h=1}^k \frac{1}{h!} + \sum_{h=k+1}^{\infty} \frac{1}{k^{h-k}} = 1 + \sum_{h=1}^k \frac{1}{h!} + \frac{1}{k-1}.$$

Note that the polynomial $p(u)$ with one variable has finitely many real roots. Hence for sufficiently large k , $p(u)$ has no root in the interval $[1 + \sum_{h=1}^k 1/h!, 1 + \sum_{h=1}^k 1/h! + 1/(k-1)]$ and so $p(e)$ has the same sign as $p(1 + \sum_{h=1}^k 1/h!)$. Clearly such a k can be effectively found. Now, given a sentence φ in $(\mathbb{R}, +, \cdot, <, 0, 1, e)$, one can apply Tarski's quantifier elimination algorithm [17] to obtain a quantifier-free sentence φ' such that φ is true iff φ' is true, and φ' is a boolean combination of formulas of the form $p(e) < 0$.

Next we show that given control states $q, q', q1, q1'$ and $\alpha \in \{\tau, \mu\}$, one can construct in $(\mathbb{R}, +, \cdot, <, 0, 1, e)$ a formula $\Phi_{q, q', q1, q1', \alpha}(V, V1)$ with free variables $V(i), V1(i), i = 1, 2, \dots, n$, that asserts $(q, V, q') \xrightarrow{\alpha}_{\mathcal{A}} (q1, V1, q1')$. In what follows, we fix $q, q', q1, q1' \in Q$ and $\alpha \in \{\tau, \mu\}$.

Clearly we can effectively compute the rational eigenvalues $\lambda_i, i = 1, 2, \dots, n$, of A_q , and for each $i = 1, 2, \dots, n$ find a rational eigenvector $f_i \in \mathbb{Q}^n$ corresponding to λ_i (i.e. $A_q \cdot f_i = \lambda_i \cdot f_i$). Let $F = (f_1 \ f_2 \ \dots \ f_n)$ be the matrix in $\mathbb{Q}^{n \times n}$ whose i -th column is f_i for $i = 1, 2, \dots, n$. From [12] it is easy to see that for a real $T \in [0, 1]$, $Val(A_q, b_q, T, V) = H(e^T)$ where $H : \mathbb{R} \rightarrow \mathbb{R}^n$ is given by

$$H(u) = F \text{diag}(u^{\lambda_1}, u^{\lambda_2}, \dots, u^{\lambda_n}) F^{-1}(V + A_q^{-1}b_q) - A_q^{-1}b_q.$$

It is easy to see that for $\alpha = \tau$, the formula $\Phi_{q, q', q1, q1', \alpha}(V, V1)$ can be constructed. The only point to note is that constants of the form e^r where $r \in \mathbb{Q}$ are definable

in $(\mathbb{R}, +, \cdot, <, 0, 1, e)$. The case $\alpha = \mu$ will follow from two observations that we now outline.

Let (q, V, q') , $(q1, V1, q1')$ be reachable configurations of \mathcal{A} such that $(q, V, q') \xrightarrow{\mu}_{\mathcal{A}} (q1, V1, q1')$. It follows from the definition of $TS_{\mathcal{A}}$ that $(q, V, q') \xrightarrow{\mu}_{\mathcal{A}} (q1, V1, q1')$ iff $V1 = Val(A_q, b_q, 1, V)$ and there exist reals t_i^{ob} in $[\delta_{ob}^0, \delta_{ob}^1]$, $i = 1, 2, \dots, n$, and a guard g such that $(q, g, q1)$ is a transition in \longrightarrow . Further, $(\langle w_1 \rangle, \langle w_2 \rangle, \dots, \langle w_n \rangle)$ satisfies g , where w_i is the i -th component of $Val(A_q, b_q, t_i^{ob}, V)$ for $i = 1, 2, \dots, n$. Firstly we note that the function $t \in [\delta_{ob}^0, \delta_{ob}^1] \rightarrow e^t \in [e^{\delta_{ob}^0}, e^{\delta_{ob}^1}]$ is continuous, increasing and onto. Thus there exist reals t_i^{ob} , $i = 1, 2, \dots, n$, satisfying the desired condition iff there exist reals $u_i \in [e^{\delta_{ob}^0}, e^{\delta_{ob}^1}]$, $i = 1, 2, \dots, n$, such that w_i is the i -th component of $H(u_i)$ for each $i = 1, 2, \dots, n$.

Secondly, we note that $-\gamma_{max} \leq w_i \leq \gamma_{max}$ for every $i = 1, 2, \dots, n$. For a guard g in \mathcal{A} , let $Valuations(g)$ be the finite set of valuations given by: (v_1, v_2, \dots, v_n) is in $Valuations(g)$ iff for each i , $v_i = m_i \epsilon$ where m_i is an integer with $\langle -\gamma_{max} \rangle \leq m_i \epsilon \leq \langle \gamma_{max} \rangle$, and (v_1, v_2, \dots, v_n) satisfies g . It follows that (w_1, w_2, \dots, w_n) satisfies g iff $(\langle w_1 \rangle, \langle w_2 \rangle, \dots, \langle w_n \rangle)$ is in $Valuations(g)$.

Putting together the above two observations, it is now clear how the formula $\Phi_{q, q', q1, q1', \mu}(V, V1)$ can be constructed. It is then also straightforward to see that given a state sequence $q_0 q_1 \dots q_\ell$ one can construct a sentence $\Phi_{q_0 q_1 \dots q_\ell}$ such that $\Phi_{q_0 q_1 \dots q_\ell}$ is true iff $q_0 q_1 \dots q_\ell$ is in $\mathcal{L}(\mathcal{A})$. \square

A consequence of theorem 1.3 is that a number of bounded model checking problems involving finite-precision hybrid automata can in principle be effectively solved.

Theorem 1.4. *There is no effective procedure which can, given a finite-precision hybrid automaton \mathcal{A} satisfying the restrictions stated in theorem 1.3 and a control state q_f of \mathcal{A} , determine whether q_f is reachable in \mathcal{A} . In other words, whether there exists a reachable configuration (q, V, q') of \mathcal{A} such that $q = q_f$.*

Proof: We shall reduce the halting problem of two-counter automata ([13]) to the control state reachability problem of the class of hybrid automata stated in the theorem.

Let $\mathcal{C} = (S, s_{in}, s_{halt}, \rightsquigarrow)$ be a two-counter automaton where S is a finite set of states, $s_{in} \in S$ the initial state, $s_{halt} \in S$ the halting state and $\rightsquigarrow \subseteq S \times \{\text{ZERO}, \text{POS}\}^2 \times \{\text{INC}, \text{DEC}\}^2 \times S$ the instruction table. The instruction $(s, O_1, O_2, \alpha_1, \alpha_2, s')$ indicates that at state s , if the sign of the integer stored in counter i is O_i then \mathcal{C} can perform action α_i (increment or decrement) on counter i and move to state s' . For example, the instruction $(s, \text{ZERO}, \text{POS}, \text{INC}, \text{DEC}, s')$ specifies that at state s , if counter 1 is zero and counter 2 is positive, then \mathcal{C} can increment counter 1, decrement counter 2 and move to state s' . The semantics of \mathcal{C} is defined in the obvious way.

In what follows, we construct a finite-precision hybrid automaton $\mathcal{A} =$

$(Q, q_{in}, V_{in}, \{\delta_{ob}^0, \delta_{ob}^1, \delta_{up}^0, \delta_{up}^1\}, \epsilon, \{\rho_q\}_{q \in Q}, \{\gamma_{min}, \gamma_{max}\}, \longrightarrow)$ over continuous variables x_1, \dots, x_n such that $\delta_{up}^0 = 0 = \delta_{up}^1$ and every ρ_q is of the form $\dot{x} = A_q x + b_q$, where A_q is a diagonalizable matrix having n distinct rational eigenvalues. Further, a designated control state $q_f \in Q$ is reachable in \mathcal{A} iff the halting state $s_{halt} \in S$ is reachable in \mathcal{C} . In fact, we will construct $\{\rho_q\}_{q \in Q}$ in such a way that every A_q is a diagonal matrix.

We set $n = 3$ and hence \mathcal{A} will be over x_1, x_2, x_3 . We first outline the construction of $Q, q_{in}, V_{in}, \{\rho_q\}_{q \in Q}, \longrightarrow$ and later discuss the choice of the parameters $\delta_{ob}^0, \delta_{ob}^1, \epsilon, \gamma_{min}, \gamma_{max}$.

The set of control states Q is $S \cup \{s_\xi^\#, s_\xi^{\#\#} \mid \xi \in \rightsquigarrow\}$ where for $\xi = (s, O_1, O_2, \alpha_1, \alpha_2, s')$ in \rightsquigarrow , $s_\xi^\# = (s, O_1, O_2, \alpha_1, \alpha_2, s', \#)$ and $s_\xi^{\#\#} = (s, O_1, O_2, \alpha_1, \alpha_2, s', \#\#)$. Intuitively, the continuous variable x_1 (x_2) will represent values of counter 1 (2). A counter having value h will be represented by the corresponding continuous variable taking the value $1 + e^{-1} + e^{-2} + \dots + e^{-h}$. In particular, a counter with value zero will be represented by the corresponding continuous variable taking the value 1.

Suppose at time T_k , the hybrid automaton \mathcal{A} is at control state s and wants to “execute” the instruction $(s, O_1, O_2, \alpha_1, \alpha_2, s')$. This is to be done by moving first to $(s, O_1, O_2, \alpha_1, \alpha_2, s', \#)$ at time T_{k+1} , and then to $(s, O_1, O_2, \alpha_1, \alpha_2, s', \#\#)$ at *exactly* time T_{k+2} , and finally to land at s' at *exactly* time T_{k+3} . In this process, the variable x_3 will be used to control that \mathcal{A} “stays” for exactly one time unit at each of $(s, O_1, O_2, \alpha_1, \alpha_2, s', \#)$, $(s, O_1, O_2, \alpha_1, \alpha_2, s', \#\#)$.

The initial control state is s_{in} . The initial valuation is $(1, 1, 1)$.

The rate functions are as follows. For $s \in S$, we set ρ_s to be $\dot{x}_1 = 0 = \dot{x}_2 = \dot{x}_3$. Suppose $(s, O_1, O_2, \alpha_1, \alpha_2, s') \in \rightsquigarrow$ is an instruction of \mathcal{C} and $step \in \{\#, \#\#\}$, then the rate function of $(s, O_1, O_2, \alpha_1, \alpha_2, s', step)$ is: $\dot{x}_1 = F_{\alpha_1}^{step}(x_1)$, $\dot{x}_2 = F_{\alpha_2}^{step}(x_2)$, $\dot{x}_3 = H^{step}(x_3)$ where:

- $F_{INC}^\#(x_i) = -x_i$ and $F_{INC}^{\#\#}(x_i) = 1$ for $i = 1, 2$.
- $F_{DEC}^\#(x_i) = -1$ and $F_{DEC}^{\#\#}(x_i) = x_i$ for $i = 1, 2$.
- $H^\#(x_3) = 1$ and $H^{\#\#}(x_3) = -1$.

The transition relation \longrightarrow of \mathcal{A} is $\bigcup_{\xi \in \rightsquigarrow} TR_\xi$, where for each $\xi = (s, O_1, O_2, \alpha_1, \alpha_2, s')$ in \rightsquigarrow , the members of TR_ξ are

$$\begin{aligned} & (s, g_\xi^s, (s, O_1, O_2, \alpha_1, \alpha_2, s', \#)) , \\ & ((s, O_1, O_2, \alpha_1, \alpha_2, s', \#), g_\xi^\#, (s, O_1, O_2, \alpha_1, \alpha_2, s', \#\#)) , \\ & ((s, O_1, O_2, \alpha_1, \alpha_2, s', \#\#), g_\xi^{\#\#}, s') , \end{aligned}$$

with the guards $g_\xi^s, g_\xi^\#, g_\xi^{\#\#}$ being specified as follows. The guard g_ξ^s is $\Phi_{O_1}(x_1) \wedge \Phi_{O_2}(x_2)$ where $\Phi_{ZERO}(x_i)$ is $x_i \leq 1$ and $\Phi_{POS}(x_i)$ is $x_i > 1$ for $i = 1, 2$. The guard $g_\xi^\#$ is $x_3 \leq 2$ and $g_\xi^{\#\#}$ is $x_3 \geq 1$.

It remains to choose the parameters $\delta_{ob}^0, \delta_{ob}^1, \epsilon, \gamma_{min}, \gamma_{max}$ appropriately. Recall that a valuation (v_1, v_2, \dots, v_n) satisfies a polynomial constraint $p(x_1, x_2,$

$\dots, x_n) < 0$ iff $p(\langle v_1 \rangle_\epsilon, \langle v_2 \rangle_\epsilon, \dots, \langle v_n \rangle_\epsilon) < 0$. Thus the main technicality is to ensure that the guards are “stable” even with finite precision measurement of values. The only restriction we need for the choice of $\delta_{ob}^0, \delta_{ob}^1, \epsilon, \gamma_{min}, \gamma_{max}$ is that ϵ integrally divides every member of $\{1, \delta_{ob}^0, \delta_{ob}^1\}$, $\langle 1 + e^{-1} \rangle_\epsilon > 1$, $\gamma_{min} \leq 1$, $\gamma_{max} \geq 2$. We emphasize that we need not demand $\delta_{ob}^0 = 1 = \delta_{ob}^1$.

It is now straightforward to establish that the halting state s_{halt} is reachable in the two-counter automaton \mathcal{C} iff the control state s_{halt} is reachable in the hybrid automaton \mathcal{A} . \square

1.6. The Perfect-Precision Setting

In this section, we discuss the results in [2, 1] regarding perfect-precision hybrid automata. Firstly, in [2] it is shown that if each continuous variable always evolves at (possibly different) constant rates in all control states and each transition guard is rectangular, then the control state sequence language of a perfect-precision hybrid automaton is regular.

Theorem 1.5 ([2]). *Let \mathcal{A} be a perfect-precision hybrid automaton such that ρ_q takes the form $\dot{x} = b_q$ (that is, A_q is the zero matrix) for each control state q , and each transition guard is rectangular. Then $\mathcal{L}(\mathcal{A})$ is a regular subset of Q^* . Further, a finite state automaton accepting $\mathcal{L}(\mathcal{A})$ can be effectively computed from \mathcal{A} .*

Proof Sketch: The proof is similar to that of theorem 1.1. Pick Δ as in the proof of theorem 1.1. Let C be the set of rational numbers c for which some transition guard of \mathcal{A} contains an inequality in one of the forms $x_i < c$, $x_i \leq c$, $x_i > c$, $x_i \geq c$. Define Γ to be the largest rational which *integrally* divides every number in the finite set of rational numbers $\{b_q(i) \cdot \Delta \mid q \in Q, i \in \{1, 2, \dots, n\}\} \cup C \cup \{\gamma_{min}, \gamma_{max}\}$. Define Θ to be the *finite* set of rational numbers $\{h\Gamma \in [-\gamma_{max}, \gamma_{max}] \mid h \in \mathbb{Z}\}$.

Now define the equivalence relation \approx over configurations in the same way as in the proof of theorem 1.1. One can then argue that \approx is a congruence relation with respect to the transition relation of $TS_{\mathcal{A}}$.

Claim 1.3. *Suppose $(q1, V1, q1') \approx (q2, V2, q2')$ and $(q1, V1, q1') \xrightarrow{\alpha}_{\mathcal{A}} (q3, V3, q3')$, then there exists a reachable configuration $(q4, V4, q4')$ such that $(q2, V2, q2') \xrightarrow{\alpha}_{\mathcal{A}} (q4, V4, q4')$ and $(q3, V3, q3') \approx (q4, V4, q4')$.*

Proof of Claim 1.3: Clearly $q1 = q2$ and $q1' = q2'$. Set $q4 = q3$ and $q4' = q3'$. Firstly by similar arguments as in the proof of claim 1.1, it is easy to see that $(q2, V2, q2')$ is a feasible configuration. We show the existence of $V4$. For brevity, we consider just the case that $\alpha = \mu$. The arguments for the case that $\alpha = \tau$ is similar but simpler.

As $\alpha = \mu$, it follows from the definition of $TS_{\mathcal{A}}$ that there exist reals t_i^{up} in $[\delta_{up}^0, \delta_{up}^1]$, $i = 1, 2, \dots, n$, such that $V3(i) = V1(i) + b_{q1'}(i) \cdot t_i^{up} + b_{q1}(i) \cdot (1 - t_i^{up})$ for

each i . Further there exist reals $t_i^{ob} \in [\delta_{ob}^0, \delta_{ob}^1]$, $i = 1, 2, \dots, n$, and a rectangular guard g such that the following conditions are satisfied: Firstly, $(q1, g, q3) \in \longrightarrow$. Secondly, the valuation U satisfies g , where $U(i) = V1(i) + b_{q1'}(i) \cdot t_i^{up} + b_{q1}(i) \cdot (t_i^{ob} - t_i^{up})$ for each i ; We shall show the existence of reals $\hat{t}_i^{up} \in [\delta_{up}^0, \delta_{up}^1]$, $\hat{t}_i^{ob} \in [\delta_{ob}^0, \delta_{ob}^1]$, $i = 1, 2, \dots, n$, such that $\|V4\| = \|V3\|$ and $\|U'\| = \|U\|$ where $V4$ is the valuation given by $V4(i) = V2(i) + b_{q1'}(i) \cdot \hat{t}_i^{up} + b_{q1}(i) \cdot (1 - \hat{t}_i^{up})$ for each i . And U' is the valuation given by $U'(i) = V2(i) + b_{q1'}(i) \cdot \hat{t}_i^{up} + b_{q1}(i) \cdot (\hat{t}_i^{ob} - \hat{t}_i^{up})$ for each i .

We note that the existence of U' satisfying $\|U'\| = \|U\|$ will guarantee that U' also satisfies g . This follows from the fact that the rational numbers appearing in g are members of Θ . So the existence of \hat{t}_i^{up} , \hat{t}_i^{ob} , $i = 1, 2, \dots, n$, suffices to establish the claim.

We fix an i in $\{1, 2, \dots, n\}$ and show the existence of \hat{t}_i^{ob} , \hat{t}_i^{up} . The arguments apply to any i . Assume $\|V3(i)\| = (\theta, \theta')$, $\|U(i)\| = (\vartheta, \vartheta')$ where $\theta, \theta', \vartheta, \vartheta' \in \Theta$ and $b_{q1'}(i) > b_{q1}(i) > 0$. Other cases can be similarly handled. For any real v , let $\Phi^\mu(v)$ be the condition

$$\begin{aligned} \exists t^{up} \in \mathbb{R}. \exists t^{ob} \in \mathbb{R}. \quad & \delta_{up}^0 \leq t^{up} \leq \delta_{up}^1 \\ & \bigwedge \theta < v + b_{q1'}(i) \cdot t^{up} + b_{q1}(i) \cdot (1 - t^{up}) < \theta' \\ & \bigwedge \delta_{ob}^0 \leq t^{ob} \leq \delta_{ob}^1 \\ & \bigwedge \vartheta < v + b_{q1'}(i) \cdot t^{up} + b_{q1}(i) \cdot (t^{ob} - t^{up}) < \vartheta' . \end{aligned}$$

It is easy to see that $\Phi^\mu(v)$ holds iff $\eta < v < \eta'$, where η is the larger of $\theta - b_{q1'}(i) \cdot \delta_{up}^1 - b_{q1}(i) \cdot (1 - \delta_{up}^1)$ and $\vartheta - b_{q1'}(i) \cdot \delta_{up}^1 - b_{q1}(i) \cdot (\delta_{ob}^1 - \delta_{up}^1)$. On the other hand, η' is the smaller of $\theta' - b_{q1'}(i) \cdot \delta_{up}^0 - b_{q1}(i) \cdot (1 - \delta_{up}^0)$ and $\vartheta' - b_{q1'}(i) \cdot \delta_{up}^0 - b_{q1}(i) \cdot (\delta_{ob}^0 - \delta_{up}^0)$. It follows that η, η' are members of Θ (if $\eta, \eta' \in [-\gamma_{max}, \gamma_{max}]$). Thus, one concludes that $\Phi^\mu(V2(i))$ holds and the existence of \hat{t}_i^{up} , \hat{t}_i^{ob} is established.

By filling in similar but simpler arguments for the case that $\alpha = \tau$, we can complete the proof of claim 1.3. \square

With claim 1.3 and similar arguments as in the proof of theorem 1.1, it is easy to show that one can effectively construct a finite automaton which accepts $\mathcal{L}(\mathcal{A})$. This completes the proof of theorem 1.5. \square

Similarly to the remark following theorem 1.1, we wish to point out that in [2], theorem 1.5 was shown for the slightly more general case where γ_{min} is allowed to be zero. This is due to that the bound γ_{min} does not play a role in settings of constant rate continuous variables.

We also showed in [1] that all of theorem 1.2, 1.3, 1.4 hold if the combination of finite precision and polynomial guards is replaced by that of perfect precision and rectangular guards (that is, every transition guard is rectangular). Before stating these results precisely, we first explain the intuition behind this observation. As noted from the proof of theorem 1.3, for a valuation V with $\gamma_{min} \leq |V(i)| \leq \gamma_{max}$ and a polynomial guard g , if $(\langle V(1) \rangle, \langle V(2) \rangle, \dots, \langle V(n) \rangle)$ satisfies g , then $(\langle V(1) \rangle, \langle V(2) \rangle, \dots, \langle V(n) \rangle)$ is one of the *finitely many* valuations (v_1, v_2, \dots, v_n)

which satisfies g and moreover fulfils the following condition: for each i , $v_i = m_i \epsilon$ where m_i is an integer with $\langle -\gamma_{max} \rangle \leq m_i \epsilon \leq \langle \gamma_{max} \rangle$. Hence, in our approaches to finite precision and polynomial guards as in the proofs of theorem 1.2, 1.3, 1.4, we are dealing with rectangular constraints involving comparisons of *actual values* of continuous variables with rational numbers of the form $m\epsilon$ where $m \in \mathbb{Z}$. This is the key observation which enables us to adapt the proofs of theorem 1.2, 1.3, 1.4, to the setting of perfect precision and rectangular guards.

Theorem 1.6. *Let \mathcal{A} be a perfect-precision hybrid automaton such that A_q is a diagonal matrix for every control state q . Further, every transition guard of \mathcal{A} is rectangular. Suppose there exists a fixed partition $\{DIF, CON\}$ of the indices $\{1, 2, \dots, n\}$ such that for each control state q , $\dot{x}_i = A_q(i, i) \cdot x_i$ if $i \in DIF$ and $\dot{x}_i = b_q(i)$ if $i \in CON$. Then $\mathcal{L}(\mathcal{A})$ is a regular subset of Q^* . Further, a finite state automaton accepting $\mathcal{L}(\mathcal{A})$ can be effectively computed from \mathcal{A} .*

Proof Sketch: The basic strategy is to extend the proof of theorem 1.5, to deal with continuous variables evolving at exponential rates. As in the proof of theorem 1.2, we shall represent exponential rates variables at the natural logarithmic scale. Due to the rectangular guards associated with transitions of the hybrid automaton and the bounds $\gamma_{min}, \gamma_{max}$, one need to handle inequalities involving comparisons of x_i with $\gamma_{min}, \gamma_{max}$, and (finitely many) rational numbers appearing in the transition guards. Thus, to partition the real line into finitely many intervals, one need to take into account *irrational* numbers of the forms $\ln \gamma_{min}, \ln \gamma_{max}, \ln c$, where c is a rational number.

For notational convenience, we assume $V_{in}(i) > 0$ for every $i \in DIF$. It will become clear that this involves no loss of generality. The key consequence of this assumption is that in any reachable configuration, the value of x_i for $i \in DIF$ will be positive.

We assume without loss of generality that for each rectangular guard g in \mathcal{A} , the valuation V satisfies g only if V is feasible.

Define Δ as in the proof of Theorem 1.1. Let C be the set of rational numbers c for which some transition guard of \mathcal{A} contains an inequality in one of the forms $x_i < c, x_i \leq c, x_i > c, x_i \geq c$. Define Γ to be the largest rational which *integrally* divides every number in the finite set of rational numbers $\{A_q(i, i) \cdot \Delta \mid q \in Q, i \in DIF\} \cup \{b_q(j) \cdot \Delta \mid q \in Q, j \in CON\} \cup \{\gamma_{min}, \gamma_{max}\} \cup C$.

Define Θ_{con} to be the finite set of rational numbers $\{h\Gamma \in [-\gamma_{max}, \gamma_{max}] \mid h \in \mathbb{Z}\}$ (where \mathbb{Z} is the set of integers). In other words, Θ_{con} contains integral multiples of Γ in the interval $[-\gamma_{max}, \gamma_{max}]$.

Let Θ_{IR} be the set of irrational numbers $\{\ln \gamma_{min}, \ln \gamma_{max}\} \cup \{\ln c \mid c \in C\}$. Define Θ_{dif} to be the *finite* set of real numbers $\{h\Gamma \in [\ln \gamma_{min}, \ln \gamma_{max}] \mid h \in \mathbb{Z}\} \cup \{\ell\Gamma + \theta \in [\ln \gamma_{min}, \ln \gamma_{max}] \mid \ell \in \mathbb{Z}, \theta \in \Theta_{IR}\}$. In other words, Θ_{dif} contains rational numbers of the form $h\Gamma$ in the interval $[\ln \gamma_{min}, \ln \gamma_{max}]$ where h is a (positive) integer, and irrational numbers of the form $\ell\Gamma + \theta$ in the interval $[\ln \gamma_{min}, \ln \gamma_{max}]$

where ℓ is an integer (that can be positive, zero or negative) and θ is a member of Θ_{IR} .

As in the proof of theorem 1.5, the set Θ_{con} (respectively Θ_{dif}) contains bounds relevant to the values of continuous variables x_i 's for $i \in CON$ (respectively $i \in DIF$). We list members of Θ_{con} in increasing order and divide the real line into finitely many open intervals and finitely many singleton intervals. Let \mathcal{I}_{con} denote the collection of these intervals. Do the same for Θ_{dif} . Let \mathcal{I}_{dif} denote the collection of intervals defined via Θ_{dif} .

Define the map $\|\cdot\|_{dif} : \mathbb{R}^+ \rightarrow \mathcal{I}_{dif}$ via: $\|v\| = I$ if $\ln v \in I$. Define $\|\cdot\|_{con} : \mathbb{R} \rightarrow \mathcal{I}_{con}$ via: $\|v\| = I$ if $v \in I$. Finally we define $\|\cdot\| : RC_{\mathcal{A}} \rightarrow (\mathcal{I}_{dif} \cup \mathcal{I}_{con})^n$ by: $\|V\| = (I_1, I_2, \dots, I_n)$ where $I_i = \|V(i)\|_{dif}$ for $i \in DIF$ and $I_i = \|V(i)\|_{con}$ for $i \in CON$. We can now define the equivalence relation $\approx \subseteq RC_{\mathcal{A}} \times RC_{\mathcal{A}}$ by: $(q1, V1, q1') \approx (q2, V2, q2')$ iff $q1 = q2$, $\|V1\| = \|V2\|$ and $q1' = q2'$.

Similarly to the proof of Claim 1.2, one can show that \approx is a congruence relation with respect to the transition relation $\Longrightarrow_{\mathcal{A}}$.

By similar arguments to the proof of theorem 1.2, it is easy to see that the equivalence classes of \approx can be effectively computed. Further, one can effectively compute the transition relation $\hookrightarrow_{\mathcal{A}}$ where, for any two equivalence classes $C1, C2$ of \approx , $C1 \xrightarrow{\alpha}_{\mathcal{A}} C2$ iff there exists (q, V, q') in $C1$, $(q1, V1, q1')$ in $C2$ such that $(q, V, q') \xrightarrow{\alpha}_{\mathcal{A}} (q1, V1, q1')$. From this, it is straightforward to construct a finite state automaton which accepts $\mathcal{L}(\mathcal{A})$. \square

Theorem 1.7. *Suppose \mathcal{A} is a perfect-precision hybrid automaton such that $\delta_{up}^0 = 0 = \delta_{up}^1$ and for every control state q , A_q is a diagonalizable matrix having n distinct rational eigenvalues. Further, every guard associated with transitions of \mathcal{A} is rectangular. Then $\mathcal{L}(\mathcal{A})$ is a recursive subset of Q^* .*

Proof Sketch: As in the proof of Theorem 1.3, the main technicality is to show that given control states $q, q', q1, q1'$ and $\alpha \in \{\tau, \mu\}$, one can construct a formula $\Phi_{q, q', q1, q1', \alpha}(V, V1)$ with free variables $V(i), V1(i)$, $i = 1, 2, \dots, n$, which asserts that $(q, V, q') \xrightarrow{\alpha}_{\mathcal{A}} (q1, V1, q1')$. And $\Phi_{q, q', q1, q1', \alpha}(V, V1)$ is a formula in $(\mathbb{R}, +, \cdot, <, 0, 1, e)$, the first order theory of reals augmented with the constant e . In what follows, we fix $q, q', q1, q1' \in Q$ and $\alpha \in \{\tau, \mu\}$.

Compute the rational eigenvalues λ_i , $i = 1, 2, \dots, n$, of A_q , and for each $i = 1, 2, \dots, n$ find a rational eigenvector $f_i \in \mathbb{Q}^n$ corresponding to λ_i (i.e. $A_q \cdot f_i = \lambda_i \cdot f_i$). Define the function H as in the proof of Theorem 1.3.

The formula $\Phi_{q, q', q1, q1', \tau}(V, V1)$ can be constructed exactly as in the proof of Theorem 1.3. We now outline the construction of $\Phi_{q, q', q1, q1', \mu}(V, V1)$.

Let (q, V, q') , $(q1, V1, q1')$ be reachable configurations of \mathcal{A} such that $(q, V, q') \xrightarrow{\mu}_{\mathcal{A}} (q1, V1, q1')$. It follows from the definition of $TS_{\mathcal{A}}$ that $(q, V, q') \xrightarrow{\mu}_{\mathcal{A}} (q1, V1, q1')$ iff $V1 = Val(A_q, b_q, 1, V)$ and there exist reals t_i^{ob} in $[\delta_{ob}^0, \delta_{ob}^1]$, $i = 1, 2, \dots, n$, and a rectangular guard g such that $(q, g, q1)$ is a transition in \longrightarrow . Further, (w_1, w_2, \dots, w_n) satisfies g , where w_i is the i -th component of

$Val(A_q, b_q, t_i^{ob}, V)$ for $i = 1, 2, \dots, n$. As in the proof of Theorem 1.3, there exist t_i^{ob} , $i = 1, 2, \dots, n$, satisfying the desired condition iff there exist reals $u_i \in [e^{\delta_{ob}^0}, e^{\delta_{ob}^1}]$, $i = 1, 2, \dots, n$, such that w_i is the i -th component of $H(u_i)$ for each $i = 1, 2, \dots, n$. It is now clear that the formula $\Phi_{q, q', q_1, q_1', \mu}(V, V1)$ can be constructed.

It follows that given a state sequence $q_0 q_1 \dots q_\ell$, one can construct a sentence $\Phi_{q_0 q_1 \dots q_\ell}$ such that $\Phi_{q_0 q_1 \dots q_\ell}$ is true iff $q_0 q_1 \dots q_\ell$ is in $\mathcal{L}(\mathcal{A})$. \square

We note that the above proof shows that Theorem 1.7 holds even if we allow polynomial guards.

Theorem 1.8. *There is no effective procedure which can, given a perfect-precision hybrid automaton \mathcal{A} satisfying the restrictions stated in Theorem 1.3 and a control state q_f of \mathcal{A} , determine whether q_f is reachable in \mathcal{A} . In other words, whether there exists a reachable configuration (q, V, q') of \mathcal{A} such that $q = q_f$.*

Proof Sketch: As in the proof of Theorem 1.4, we reduce the halting problem of two-counter automata ([13]) to the control state reachability problem of the class of hybrid automata stated in the Theorem 1.8.

Let $\mathcal{C} = (S, s_{in}, s_{halt}, \rightsquigarrow)$ be a two-counter automaton. We construct a perfect-precision hybrid automaton $\mathcal{A} = (Q, q_{in}, V_{in}, \{\delta_{ob}^0, \delta_{ob}^1, \delta_{up}^0, \delta_{up}^1\}, \{\rho_q\}_{q \in Q}, \{\gamma_{min}, \gamma_{max}\}, \longrightarrow)$ over continuous variables x_1, \dots, x_n such that $\delta_{up}^0 = 0 = \delta_{up}^1$ and every ρ_q is of the form $\dot{x} = A_q x + b_q$, where A_q is a diagonalizable matrix having n distinct rational eigenvalues. Further, every transition guard of \mathcal{A} is rectangular. Lastly, a designated control state $q_f \in Q$ is reachable in \mathcal{A} iff the halting state $s_{halt} \in S$ is reachable in \mathcal{C} .

We set $n = 3$ and hence \mathcal{A} will be over x_1, x_2, x_3 . We construct $Q, q_{in}, V_{in}, \{\rho_q\}_{q \in Q}, \longrightarrow$ exactly as in the proof of Theorem 1.4. As for the choice of the parameters $\delta_{ob}^0, \delta_{ob}^1, \gamma_{min}, \gamma_{max}$, we need only require that $\gamma_{min} \leq 1$ and $\gamma_{max} \geq 2$. We emphasize that we need not demand $\delta_{ob}^0 = 1 = \delta_{ob}^1$.

It is now straightforward to establish that the halting state s_{halt} is reachable in the two-counter automaton \mathcal{C} iff the control state s_{halt} is reachable in \mathcal{A} . \square

1.7. Discussion

We have shown here that the twin features of discrete time and finite precision allows to deal with hybrid automata whose dynamics are governed by restricted linear differential operators and whose transitions have polynomial guards. Indeed, their control state sequence languages will be regular and these state sequence languages can be effectively constructed although we do not impose the reset restriction.

Our results seem to suggest that once observational and update delays are included to further reduce the expressive power of these automata, one may be able to handle much richer continuous dynamics. The key obstacle here is our lack

of techniques for constructing effective rational approximations of the continuous dynamics. Here, the mathematical foundations provided in [5] and the logical underpinnings developed in [15, 16] may provide a sound basis.

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