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Schnorr trivial sets and truth-table reducibility

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T e c h n i c a l R e p o r t

Foreword

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Abstract

In this paper, we give several characterizations of Schnorr trivial sets, including a new lowness notion for Schnorr triviality based on truth-table reducibility. These characterizations enable us to see not only that some natural classes of sets, including maximal sets, are composed entirely of Schnorr trivials, but also that the Schnorr trivial sets form an ideal in the truth-table degrees but not the weak truth-table degrees. This answers a question of Downey, Griffiths and LaForte.

1 Introduction

One of the major achievements in the study of Martin-Löf randomness is Nies's discovery [14] that the following conditions on a set A coincide.

- A is low for Martin-Löf randomness, that is, every set that is Martin-Löf random unrelativized is also Martin-Löf random relative to A.
- A is low for prefix-free Kolmogorov complexity, that is, the prefix-free Kolmogorov complexity relative to A differs from its unrelativized version only by a constant.
- A is trivial for prefix-free Kolmogorov complexity, that is, the complexity of $A(0)A(1)\ldots A(n)$ is not greater than the complexity of 0^n plus some constant.

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- A is a basis for Martin-Löf randomness, that is, A is Turing reducible to a set which is Martin-Löf random relative to A.

Furthermore, Nies showed in the same paper that these sets form an ideal with respect to Turing reducibility.

Schnorr [19, 20] introduced an alternative definition of randomness, as he felt that Martin-Löf's definition [11] was too restrictive. For example, Martin-Löf random sets do not exist in any incomplete r.e. Turing degree [1], while Schnorr random sets exist in every high r.e. Turing degree [15].

It is possible to define trivial sets not only for Martin-Löf randomness but also for Schnorr randomness [3, 5, 6]. This notion turned out to be incompatible with Turing reducibility, since the Schnorr trivials are not closed downward under this reduction [3]. Furthermore, when defining lowness for Schnorr randomness using Turing reducibility, only the Schnorr trivial sets of hyperimmune-free Turing degree are low for Schnorr randomness [7]. Here, as well, Turing reducibility does not seem to be appropriate.

In this paper, the problem of finding results analogous to those of Nies is reconsidered. Our main contribution is to consider truth-table reducibility instead of Turing reducibility. In Section 2, we discuss how to define a set as "truth-table low for Schnorr randomness". We first need to make the notion of being Schnorr random relative to A sensitive to truth-table reducibility. Otherwise, such a characterization could not be expected. In Section 3, we present a large number of characterizations of Schnorr triviality. Among these, the most striking one is the coincidence of Schnorr triviality with the notion of being truth-table low for Schnorr random. In Section 4, a proof is given for the result from Franklin's thesis [5] that the Schnorr trivial sets form a truth-table ideal; this result answered an open question of Downey, Griffiths and LaForte [3]. Furthermore, we show that Schnorr trivial sets are not closed under more general reducibilities like weak truth-table reducibility. Therefore, truth-table reducibility is the appropriate reducibility for the study of Schnorr triviality. In Section 5, we show that all maximal sets are Schnorr trivial and that this observation cannot be generalized to r-maximal or cohesive sets. In Section 6, we show that the Schnorr trivial sets cannot be characterized as bases of Schnorr randomness with respect to truth-table reducibility.

The interested reader is referred to the textbooks of Odifreddi [16, 17], Rogers [18] and Soare [21] for background information on recursion theory. Li and Vitányi [10], Schnorr [19] and Downey, Hirschfeldt, Nies and Terwijn [4] provide an overview of algorithmic randomness. However, before we begin, we recall the martingale equivalents of Schnorr randomness and recursive randomness from [19] as well as a definition of Schnorr triviality that is easily seen to be equivalent to the original in [3].

Theorem 1.1. *A set R is Schnorr random if there is no recursive martingale mg and unbounded, nondecreasing recursive function f such that $mg(R(0)R(1)\dots R(n)) \geq f(n)$ for infinitely many n.*

Theorem 1.2. *A set R is recursively random if there is no recursive martingale such that for any m, there is an n such that $mg(R(0)R(1)\dots R(n)) \geq m$.*

Definition 1.3. *A set A is Schnorr trivial if for every prefix-free machine M with $\sum_{\sigma\in\mathrm{dom}(M)} 2^{-|\sigma|} = 1$, there are a prefix-free machine N and a constant c such that*

- $\sum_{\sigma\in\mathrm{dom}(N)} 2^{-|\sigma|} = 1$ *and*
- *for all n and all $\sigma \in \mathrm{dom}(M)$ with $M(\sigma) = 0^n$ there is a $\tau \in \mathrm{dom}(N)$ such that $N(\tau) = A(0)A(1)\dots A(n)$ and $|\tau| \leq |\sigma| + c$.*

2 Relativizing Schnorr randomness

Before relativizing the notion of Schnorr randomness with respect to truth-table reducibility, we give the characterization in Proposition 2.1 below. We then discuss which of these notions is most suitable for relativization. The main elements of the proof of the proposition are two aspects of the martingale: the "savings property" [12] and how the performance bound of the martingale is introduced.

Recall that a recursive martingale mg is a recursive function mapping strings in $\{0,1\}^*$ to nonnegative recursive real numbers that satisfies the equation

$$\forall\sigma \in \{0,1\}^* \, [mg(\sigma) = (mg(\sigma 0) + mg(\sigma 1))/2].$$

Proposition 2.1. *The following conditions on a set R are equivalent to "R is not Schnorr random":*

1. *There is a recursive martingale mg and a recursive function f such that the relation $mg(R(0)R(1)\dots R(f(n))) \geq n$ holds for infinitely many n.*
2. *There is a recursive martingale mg' and a recursive function f' such that the relation $mg'(\sigma\tau) + 2 \geq mg'(\sigma)$ holds for all $\sigma, \tau \in \{0,1\}^*$ and the relation $mg'(R(0)R(1)\dots R(f'(n))) \geq n$ holds for infinitely many n.*
3. *There is a recursive martingale mg'' and an unbounded and nondecreasing recursive function f'' such that $mg''(R(0)R(1)\dots R(n)) \geq f''(n)$ for infinitely many n.*

Proof. Note that the third condition is equivalent to "R is not Schnorr random". Hence we just prove that all three conditions are equivalent.

To see that the first two are equivalent, it is enough to show that the first implies the second. We assume without loss of generality that mg never takes a power of 2 as its value and that the initial value of mg is a recursive real number strictly between 0 and 1. Otherwise, the following case distinction might be nonrecursive due to infinite running time. Now we define inductively a sequence $mg_0, mg_1, \ldots$ of recursive martingales with the same initial value as mg that satisfy the following rule for $a \in \{0,1\}$:

$$mg_k(\sigma a) = \begin{cases} mg(\sigma a) & \text{if } mg_k(\sigma) < 2^k; \\ mg_k(\sigma) & \text{if } mg_k(\sigma) \geq 2^k. \end{cases}$$

Then we let

$$mg'(\sigma) = \sum_{k=0,1,2,\ldots} 2^{-k} \cdot mg_k(\sigma)$$

for all $\sigma \in \{0,1\}^*$. Furthermore, we define $f'(n) = \max\{f(m) : m \leq 2^{n+1}\}$. This translation of the bounds is based on the observation that

$$\exists^\infty n \, \exists m \in \{2^n, 2^n+1, \ldots, 2^{n+1}-1\} \, [mg(R(0)R(1)\ldots R(f(m))) \geq m].$$

By induction over the definition of mg_k, we can show that for infinitely many n, there is some $m < 2^{n+1}$ such that for all $k \leq n$, the inequalities

$$mg_k(R(0)R(1)\ldots R(f(m))) \geq 2^k$$

and

$$mg_k(R(0)R(1)\ldots R(f'(n))) \geq 2^k$$

hold. It follows that

$$mg'(R(0)R(1)\ldots R(f'(n))) \geq \sum_{k \leq n} 2^{-k} mg_k(R(0)R(1)\ldots R(f'(n))) \geq \sum_{k \leq n} 2^{-k} \cdot 2^k \geq n$$

for all but finitely many n. Now we only need to show that

$$\forall \sigma, \tau \in \{0,1\}^* \, [mg'(\sigma\tau) + 2 \geq mg'(\sigma)].$$

Given any σ and τ, choose n to be the maximal integer such that $2^n \leq mg(\sigma)$. For $k > n$ the martingale mg_k still follows mg, while for $k \leq n$, the martingale mg_k is already constant above σ. Then the following two equations hold.

$$\begin{aligned} mg'(\sigma) &= \sum_{k \leq n} 2^{-k} mg_k(\sigma) + \sum_{k > n} 2^{-k} \cdot mg(\sigma) \\ mg'(\sigma\tau) &= \sum_{k \leq n} 2^{-k} mg_k(\sigma\tau) + \sum_{k > n} 2^{-k} \cdot mg(\sigma\tau) \end{aligned}$$

The second term in the formula for $mg'(\sigma)$ is bounded by 2. This is the only part that can become lost when considering $\sigma\tau$ instead of σ, so since n was chosen such that $\sum_{k\leq n} 2^{-k} mg_k(\sigma\tau) = \sum_{k\leq n} 2^{-k} mg_k(\sigma)$, the condition

$$\forall \sigma, \tau \in \{0,1\}^* \, [mg'(\sigma\tau) + 2 \geq mg'(\sigma)]$$

is indeed satisfied. This completes the proof for the implication from the first to the second condition.

To prove that the second condition implies the third, we first show that f' can be assumed to be strictly monotonically increasing. Given a recursive function g', one can define $f'(0) = g'(0) + g'(1) + g'(2)$ and $f'(n+1) = g'(n+3) + f'(n) + 1$ inductively. There are infinitely many n such that $mg'(R(0)R(1)\ldots R(g'(n+2))) \geq n+2$, so it follows that $mg'(R(0)R(1)\ldots R(f'(n))) \geq n$ from mg''s properties and the fact that $f'(n) \geq g'(n+2)$. Now let $f''(n) = \max\{m : m = 0 \vee f'(m) \leq n\}$ for all n. As there are infinitely many m such that

$$mg'(R(0)R(1)\ldots R(f'(m))) \geq m,$$

we take the value $n = f'(m)$ for these m and see that

$$mg''(R(0)R(1)\ldots R(n)) \geq f''(n)$$

with $mg'' = mg'$. Hence the third condition follows from the second.

To see that the second condition follows from the third, assume that mg'' and f'' are given and that mg' is built from mg'' as mg' was built from mg above. Furthermore, for any m, let

$$f(m) = \min\{n : f''(n) > 2^{m+4}\}$$

and note that this minimum exists by the unboundedness of f''. Furthermore, for all $\sigma, \tau \in \{0,1\}^*$, $mg'(\sigma\tau) \geq m$ whenever $mg''(\sigma) \geq 2^m$. There are infinitely many m for which there is an $n \in \{f(m), f(m)+1, \ldots, f(m+1)-1\}$ such that $mg''(R(0)R(1)\ldots, R(n)) \geq f''(n)$. It follows that $mg'(R(0)R(1)\ldots R(n)\tau) \geq m$ for all τ and, in particular, $mg'(R(0)R(1)\ldots R(f'(m))) \geq m$. ∎

This characterization of non-Schnorr randomness is influenced by the goal of establishing a link between the Schnorr triviality of A and the sets that are Schnorr random relative to A in the context of truth-table reducibility. First, we show that the third characterization in Proposition 2.1 does not work.

Theorem 2.2. *There is a Schnorr random set $R \leq_T K$ and a recursive martingale mg such that for every $A \geq_T K$, there is a nondecreasing unbounded function $lb \leq_{tt} A$ such that $mg(R(0)R(1)\ldots R(n)) \geq lb(n)$ for all n.*

Proof. Nies, Terwijn and Stephan [15] showed that there is a set $R \equiv_T K$ which is Schnorr random but not recursively random. Hence there is a recursive martingale mg which succeeds on R, although without the bound constraints given by the martingale definition of Schnorr randomness. It was shown in Proposition 2.1 that we can replace mg by another recursive martingale mg' such that $mg'(R(0)R(1)\dots R(m)) \geq mg'(R(0)R(1)\dots R(n))-2$ for all n and all $m > n$. This transformation also preserves the fact that $mg' \leq_{tt} K$. Now let $lb(n)$ be the largest natural number k such that either $k = 0$ or there is a number $m \leq n$ for which $R(0)R(1)\dots R(m)$ can be computed within m steps from A (which can be evaluated within a truth-table reduction) and $mg'(R(0)R(1)\dots R(m)) \geq k+2$. It is easy to see that $mg'(R(0)R(1)\dots R(n)) \geq lb(n)$ for all n. Furthermore, lb is nondecreasing by definition. It is also easy to see that lb is unbounded, as mg' takes arbitrarily large values on R. Furthermore, $R \leq_T A$. ∎

This indicates that the following definition is a more suitable candidate for properties that we will call "truth-table Schnorr randomness" and "truth-table low for Schnorr randomness".

Definition 2.3. A set R is truth-table Schnorr random relative to A if and only if there is no martingale $mg \leq_{tt} A$ and no recursive bound function b such that

$$\exists^\infty n\,[mg(R(0)R(1)\dots R(b(n))) \geq n].$$

A set A is truth-table low for Schnorr randomness if every Schnorr random set R is truth-table Schnorr random relative to A.

Remark 2.4. Note that formally we would have to require that $b \leq_{tt} A$, but we can show that this is not necessary. The proof of the second condition from the first in Proposition 2.1 preserves being truth-table reducible to some given set; therefore the property can be assumed without loss of generality. For any bound b, there are only finitely many choices which can be computed without knowing the oracle A. Hence, given a bound $b' \leq_{tt} A$, we can obtain a new recursive bound $b(n)$ by taking the maximum of all possible choices for $b'(4n+4)$. There are infinitely many n for which there is an $m \in \{4n, 4n+1, 4n+2, 4n+3\}$ such that $mg(R(0)R(1)\dots R(b'(m))) \geq m$, so $mg(R(0)R(1)\dots R(b(n))) \geq n$ for these n as well. Thus mg together with b witnesses that R is not truth-table Schnorr random relative to A. Thus, it is sufficient to assume that the bound function b is recursive.

3 Characterizing Schnorr triviality

The goal of this section is to give several equivalent characterizations of Schnorr triviality. The following theorem, which states that every set A must either truth-

table compute a martingale which succeeds on some recursively random set in the sense of Schnorr or be "captured" in a particular way by a recursive function, is an important preliminary result. We will show later that the second condition in the following theorem is equivalent to Schnorr triviality.

Theorem 3.1. *For every set A, exactly one of the following two conditions hold.*

1. *There is a recursively random set R that is not truth-table Schnorr random relative to A.*
2. *For every recursive function u, there is a recursive function g such that*

$$A(0)A(1)\dots A(u(n)) \in \{g(0), g(1), \dots, g(16^n)\}$$

for almost all n.

Proof. We will show that for every recursive function u, either the function g exists or a set $R \leq_T A''$ can be constructed which is recursively random but not truth-table Schnorr random relative to A. It is sufficient to consider only functions u that are strictly increasing.

Construction. Let $mg_1, mg_2, \dots$ be a listing of all recursive martingales which only assume positive values. It is clear that it is enough to diagonalize against these martingales. Each martingale is intended to be incorporated into the martingale that we are constructing, mg', at some level c_k.

Furthermore, let $J_n = \{0,1\}^{u(n)+1}$ and partition the natural numbers into intervals I_σ with σ ranging over $\bigcup_n J_n$ such that I_σ has $2n$ elements for every $\sigma \in J_n$. Now we inductively define a martingale mg' and a set R as follows.

At the beginning, we have $k = 0$ and $mg'(\tau) = 1$ for the empty string τ. For each σ, let n be the unique number such that $\sigma \in J_n$ and let k be the largest index of the martingales used thus far. Let $m = \min(I_\sigma)$. Note that $c_1, c_2, \dots, c_k$ are already defined and take values between 0 and m. We will construct a martingale mg' such that for all $\tau \in \{0,1\}^{m+1} \cup \{0,1\}^{m+2} \cup \dots \cup \{0,1\}^{m+2n}$, the equation

$$mg'(\tau) = 2^{-k} + \sum_{d \in \{1,2,\dots,k\}} 2^{-d} \cdot \frac{mg_d(\tau)}{mg_d(\tau(0)\tau(1)\dots\tau(c_d))}$$

holds. If $k = 0$, this just means that $mg'(\tau) = 1$. We may observe that this equation will be passed on as an invariant at each stage of the construction.

Having extended mg', we now check to see which of the following three cases holds.

Case 1: $\sigma \neq A(0)A(1)\ldots A(u(n))$.

Case 2: $\sigma = A(0)A(1)\ldots A(u(n))$ and
$mg'(R(0)R(1)\ldots R(m)1^{2n}) \leq mg'(R(0)R(1)\ldots R(m)) + 2^{-n}$.

Case 3: $\sigma = A(0)A(1)\ldots A(u(n))$ and
$mg'(R(0)R(1)\ldots R(m)1^{2n}) > mg'(R(0)R(1)\ldots R(m)) + 2^{-n}$.

In Cases 1 and 3, we define R on I_σ to be such that $mg'(R(0)R(1)\ldots R(m+2n))$ is as small as possible. In Case 1, we can see that this will guarantee that

$$mg'(R(0)R(1)\ldots R(m+2n)) \leq mg'(R(0)R(1)\ldots R(m)).$$

In Case 3, the 2^{-n} gain on the sequence 1^{2n} must be compensated by a loss of at least 8^{-n} on at least one of the 4^n other possible extensions of R on these $2n$ bits, so

$$mg'(R(0)R(1)\ldots R(m+2n)) \leq mg'(R(0)R(1)\ldots R(m)) - 8^{-n}.$$

Let k remain unchanged in both cases.

In Case 2, let $R(m+i) = 1$ for $i \in \{1, 2, \ldots, 2n\}$. Furthermore, choose $c_{k+1} = m + 2n$ such that

$$mg'(\tau) = 2^{-k-1} + \sum_{d \in \{1,2,\ldots,k,k+1\}} 2^{-d} \cdot \tfrac{mg_d(\tau)}{mg_d(\tau(0)\tau(1)\ldots\tau(c_d))}$$

for all $\tau \in \{0,1\}^{m+2n}$. Note that

$$\forall \tau \in \{0,1\}^{m+2n+1} \left[\tfrac{mg_{k+1}(\tau)}{mg_{k+1}(\tau(0)\tau(1)\ldots\tau(c_{k+1}))} = 1\right]$$

and therefore the new definition and the old definition of mg' on level $m+2n$ give the same value. We have now incorporated another recursive martingale into mg'.

Verification. Now we must verify this construction. First, we consider the case in which Case 2 occurs infinitely often in the construction. In this case, every martingale mg_k is incorporated on some level c_k into the construction and

$$mg'(R(0)R(1)\ldots R(m)) = 2^{-k}\tfrac{mg_k(R(0)R(1)\ldots R(m))}{mg_k(R(0)R(1)\ldots R(c_k))} + mg''(R(0)R(1)\ldots R(m))$$

for some martingale mg'' and all $m \geq c_k$. It follows that if mg_k witnesses that R is not recursively random, so does mg'. However, by construction, mg' gains at most 2^{-n} in Case 2 and loses some value in Case 1 and Case 3. Therefore, mg' does not demonstrate that R is not recursively random.

On the other hand, there would be infinitely many n such that $I_{A(0)A(1)\ldots A(f(n))} \subseteq R$.

This means that a martingale $mg \leq_{tt} A$ that splits the capital into pieces of size 2^{-n-1} in the beginning and uses each such piece to bet $2n$ times that all the members of $I_{A(0)A(1)\ldots A(u(n))}$ are in R can be constructed. This results in a capital of 2^{n-1} if it succeeds. It follows that there would be infinitely many n such that this succeeds and that the martingale mg would witness that R is not truth-table Schnorr random relative to A.

Now we consider the case in which Case 2 occurs only finitely often in the construction. For almost all n and $\sigma = A(0)A(1)\ldots A(u(n))$, the construction runs through Case 3 and the capital of 8^{-n} is lost. Furthermore, whenever the construction runs through Case 1, the capital does not increase. Then the capital has a peak value r at some stage. It follows that for almost all n, there are only $r \cdot 8^n$ many strings $\sigma \in J_n$ for which at least 8^{-n} is lost while following R. Furthermore, as the algorithm runs through Case 1 and Case 3 all but finitely many times, both the martingale mg' and the set R are recursive. Hence these $r \cdot 8^n$ strings $\sigma \in J_n$ on which the capital decreases by at least 8^{-n} can be computed and there is a recursive function g such that for almost all n, there is a number $\ell \in \{16^{n-1}+1, 16^{n-1}+2, \ldots, 16^n\}$ with $g(\ell) = \sigma$ for each such string σ. It follows that

$$A(0)A(1)\ldots A(u(n)) \in \{g(0), g(1), \ldots, g(16^n)\}$$

for almost all n. ∎

We now present the main theorem.

Theorem 3.2. *The following conditions are equivalent.*

(a) *A is Schnorr trivial.*

(b) *For all recursive probability distributions μ on $\{0,1\}^*$, there is a further recursive probability distribution ν on $\{0,1\}^*$ and a rational constant $q > 0$ satisfying the inequality $\nu(\{A(0)A(1)\ldots A(n)\}) \geq q \cdot \mu(\{0,1\}^{n+1})$ for all n.*

(c) *There is a recursive function h such that for all $f \leq_{tt} A$, there is a recursive function g such that $f(n) \in \{g(0), g(1)\ldots, g(h(n))\}$ for almost all n.*

(d) *For every recursive nondecreasing and unbounded function h and every $f \leq_{tt} A$, there is a recursive function g such that $\forall n\, \exists m \leq h(n)\, [f(n) = g(\langle n, m\rangle)]$.*

(e) *For every martingale $mg \leq_{tt} A$ and every recursive function u, there is a recursive martingale mg' such that for all sets B, for all $n > 0$ and for all $m \leq u(n)$, the inequality $mg'(B(0)B(1)\ldots B(m)) \cdot n \geq mg(B(0)B(1)\ldots B(m))$ holds.*

(f) *No martingale $mg \leq_{tt} A$ succeeds in the sense of Schnorr on any Schnorr random set; that is, A is tt-low for Schnorr randomness.*

Proof. **(a)** implies **(b)**: The only work required is to replace the prefix-free machines by recursive measures. To see this, note that for every string θ, the number

$$\mu(\{\theta\}) = \sum_{\sigma:M(\sigma)\downarrow=\theta} 2^{-|\sigma|} = 1 - \sum_{\sigma:M(\sigma)\downarrow\neq\theta} 2^{-|\sigma|}$$

is a recursive real number since it can be approximated from below and above. The measures μ and ν will take the roles of M and N, respectively, and the constant q replaces the c. Note that one can obtain rational-valued measures that take only positive values by dividing the measure by 2, adding 2^{-n-2} for the n-th string and then modifying the resulting measure so that the sum is still 1 and each value is rational. Note that each value is changed by a factor less than one half.

(b) implies **(c)**: Let $h(n) = 3^n$. Assume that $f \leq_{tt} A$ is given via a tt-reduction φ_e and let $u(n)$ be the use function for $f(n)$. Without loss of generality, u is not only recursive but strictly increasing. Now define μ such that $\mu(\{0,1\}^{u(n)+1}) = 2^{-n-1}$ for all n. By (b), there is another measure ν and a constant q such that $\nu(\{A(0)A(1)\ldots A(u(n))\}) \geq q \cdot 2^{-n-1}$. Now we define the function g such that $\{g(3^{n-1}), g(3^{n-1}+1), \ldots, g(3^n-1)\}$ contains all numbers m such that $m = \varphi_e^\sigma(n)$ and $\nu(\{\sigma\}) > 3^{1-n}$. Note that such a function g can be constructed, as there are at most 3^{n-1} possible choices for σ for each n, and in our construction, we may have $2 \cdot 3^{n-1}$ such σ for each n. Furthermore, $\nu(f(n)) > q \cdot 2^{-n-1}$ and thus $\nu(f(n)) > 3^{1-n}$ for almost all n. Hence g is as required.

(c) implies **(d)**: Let h be the nondecreasing, unbounded and recursive function of (d) and let h' be the corresponding recursive bound assumed to exist by condition (c). Let $f \leq_{tt} A$ be given. Now define a function u to translate the bound h' into h as below:

$$u(n) = \min\{m : h(m) > h'(n+1)\}.$$

The idea is not to consider f itself but strings of values of f of length $u(n)$ so that any information provided through condition (c) is actually providing information for all values $f(m)$ with $m \leq u(n)$ and not just for a few values of f. These strings can also be computed via a truth-table reduction to A, so by (c), there is a recursive function g' such that

$$\forall^\infty n\,[f(0)f(1)\ldots f(u(n)) \in \{g'(0), g'(1), \ldots, g'(h(n))\}].$$

This permits us to define $g(\langle m,n\rangle)$ as component n of $g'(m)$ whenever $g'(m)$ is a string of length at least n and 0 otherwise. The function u is defined such that for every n there is a k with $h'(k) < h(n)$ and $u(k) \geq n$. It follows that $f(0)f(1)\ldots f(u(k))$ occurs among the first $h(n)$ values of g' and $f(n) \in \{g(\langle n,0\rangle), g(\langle n,1\rangle), \ldots, g(\langle n,h(n)\rangle)\}$.

(d) implies **(e)**: Assume that $mg \leq_{tt} A$ and u is a recursive and strictly increasing function. The goal is to construct a recursive martingale mg' such that mg and mg' differ at most by a factor of n on input shorter than $u(n)$. To do this, let $h(n) = 2^n$ and let $f(n)$ be a representation of the martingale mg_n that satisfies the condition

$$\forall \sigma \in \{0,1\}^{u(4^{n+1})+1} \, \forall \tau \in \{0,1\}^* \, [mg_n(\sigma\tau) = mg(\sigma)]$$

and has initial value 1. This martingale mg_n is unique. There is a function g which takes representations of martingales as values such that the martingale $mg'_{\langle n,m \rangle}$ given by $g(\langle n, m \rangle)$ is recursive and for every n there is an $m < 2^n$ such that $mg'_{\langle n,m \rangle} = mg_n$. Now let

$$mg'(\sigma) = \sum_{n>0} \sum_{m<2^n} 4^{-n} mg'_{\langle n,m \rangle}(\sigma).$$

It is easy to see that mg' is a recursive function that satisfies the equation $mg'(\sigma) = \frac{1}{2}(mg'(\sigma 0) + mg'(\sigma 1))$. Furthermore, the initial value of mg' is $\sum_{n>0} \sum_{m<2^n} 4^{-n} \cdot 1 = \sum_{n>0} 2^n \cdot 4^{-n} = \sum_{n>0} 2^{-n} = 1$. Hence mg' is a recursive martingale. It is also true that

$$\forall \sigma \in \{0,1\}^* \, \forall n > 0 \, [|\sigma| \leq u(4^{n+1}) + 1 \Rightarrow mg'(\sigma) \geq 4^{-n} mg(\sigma)],$$

as there is some $m < 2^n$ with $mg(\sigma) = mg'_{\langle m,n \rangle}(\sigma)$ and the weight of $mg'_{\langle m,n \rangle}(\sigma)$ in the sum is 4^{-n}. It follows that for each $m \in \{4^n, 4^n + 1, \ldots, 4^{n+1} - 1\}$ and each σ with length at most $u(m)$, the inequality $mg'(\sigma) \cdot m \geq mg(\sigma)$ holds.

(e) implies **(f)**: Assume that R is a set which is not truth-table Schnorr random relative to A. Then there is a martingale $mg \leq_{tt} A$ and a recursive bound function b such that

$$\exists^\infty n \, [mg(R(0)R(1) \ldots R(b(n))) \geq n].$$

Let $u(n) = b(4n^2)$. By assumption, there is a recursive martingale mg' such that

$$\forall n \, [mg'(R(0)R(1) \ldots R(u(n))) \cdot n \geq mg(R(0)R(1) \ldots R(u(n)))].$$

Furthermore, there are infinitely many n such that there is an $m \in \{4n^2, 4n^2 + 1, \ldots, 4(n+1)^2 - 1\}$ such that $mg(R(0)R(1) \ldots R(b(m))) \geq m \geq 4n^2$. It follows that if such an m exists for n, the inequality $mg(R(0)R(1) \ldots R(b(4(n+1)^2))) \geq n^2$ holds. Since $u(n+1) = b(4(n+1)^2)$, we can deduce that

$$\exists^\infty n \, [mg'(R(0)R(1) \ldots R(u(n))) \geq n]$$

and we are done.

(f) implies **(c)**: Here we define $h(n) = 16^n$ and use Theorem 3.1.

(c) implies **(b)**: We use the fact, proved above, that (c) implies (d) and prove that (d) implies (b). Let μ be any recursive distribution on $\{0,1\}^*$. Without loss of generality, $\mu(\sigma) > 0$ for all σ. There is a strictly increasing recursive function f such that

$$\forall n\,[\mu(\{\sigma : |\sigma| \geq f(n)\}) \leq 2^{-n}].$$

By (d), there is a recursive function g such that

$$\forall n\,\forall m \leq f(n)\,[A(0)A(1)\ldots A(m) \in \{g(\langle m,0\rangle), g(\langle m,1\rangle), \ldots, g(\langle m,n\rangle)\}].$$

Without loss of generality, we can assume that for every m and k, the value of $g(\langle m,k\rangle)$ is a string of length $m+1$. It follows from the choice of f that

$$\sum_{k\in\{f(n),f(n)+1,\ldots,f(n+1)-1\}} \mu(\{0,1\}^{k+1}) \leq 2^{-n}$$

and that

$$\sum_{k\in\{f(n),f(n)+1,\ldots,f(n+1)-1\}} \mu(\{0,1\}^{k+1})\cdot(n+1) \leq (n+1)\cdot 2^{-n}.$$

Now we use the fact that $\sum_n (n+1)\cdot 2^{-n} = 4$ to see that for each n and each length $k \in \{f(n), f(n)+1, \ldots, f(n+1)-1\}$, there are $n+1$ candidate strings of length k which receive measure $0.1\cdot\mu(\{0,1\})^{k+1}$ and that the sum over the assigned measure is bounded by $0.1\cdot(n+1)\cdot 2^{-n}$ for the part belonging to n and by 0.4 if summed over all n. We can distribute the remaining measure of about 0.6 in such a way that we obtain a measure ν such that

$$\forall n\,[\nu(\{\sigma : f(n) \leq |\sigma| < f(n+1)\}) \geq 0.25\cdot(n+1)\cdot 2^{-n}]$$

and

$$\forall n \forall m \leq f(n)\,[\nu(g(\langle m,n\rangle)) \geq 0.1\cdot\mu(\{0,1\}^{m+1})].$$

It follows that

$$\forall m\,[\nu(\{A(0)A(1)\ldots A(m)\}) \geq 0.1\cdot\mu(\{0,1\}^{m+1})],$$

so ν has the required properties.

(b) implies **(a)**: This implication follows standard arguments with Kraft-Chaitin sets; see, for example, Franklin's work on Schnorr trivial sets [6]. We take the machine M from condition (a) and consider the corresponding measure μ defined by M. This lets us conclude that there is another measure ν with the properties stated, so we can then construct N using the Kraft-Chaitin Theorem. ∎

Remark 3.3. *As truth-table reductions depend on the values of the oracle below the use, we can easily get an additional characterization. Let h be a superlinear recursive function (that is, for all c and infinitely many n, $h(n) > c \cdot n$) such that $h(n+1) > h(n)$ and $h(n+2) - h(n+1) \geq h(n+1) - h(n)$ for all n. Then a set A is Schnorr trivial if and only if for every recursive function u, there is a recursive function g such that*

$$\forall n \, [A(0)A(1) \ldots A(u(n)) \in \{g(h(n)), g(h(n)+1), \ldots, g(h(n+1)-1)\}].$$

The most common choice for h is $h(n) = 2^n$.

4 Reducibilities

The first application of these characterizations is a straightforward proof that the class of Schnorr trivial sets is a truth-table ideal. Downey, Griffiths and LaForte [3] proved that Schnorr trivials are closed downwards under truth-table reduction and asked whether they are also closed under join. Franklin [5] gave a positive answer to this question. We prove the following theorem using the above characterizations as in [5].

Theorem 4.1 [3, 5]**.** *Let A and B be Schnorr trivial and let $C \leq_{tt} A \oplus B$. Then C is Schnorr trivial.*

Proof. Let $h(n) = 4^n$ and assume that $f \leq_{tt} C$. Then $f \leq_{tt} A \oplus B$. Let u be the use function for f. By Remark 3.3, there are recursive functions g_A and g_B such that the following conditions hold for all n.

$$\begin{aligned} A(0)A(1) \ldots A(u(n)) &\in \{g_A(2^n), g_A(2^n+1), \ldots, g_A(2^{n+1}-1)\} \\ B(0)B(1) \ldots B(u(n)) &\in \{g_B(2^n), g_B(2^n+1), \ldots, g_B(2^{n+1}-1)\} \end{aligned}$$

It follows that one can compute $f(n)$ from a pair $\langle g_A(i), g_B(j) \rangle$ such that $i, j < 2^n$ where the computation also terminates for false pairs, although probably with false values. There are 4^n such pairs and each of these pairs produces one candidate. Hence we can construct a function g that lists exactly 4^n candidates for each n, including $f(n)$. It follows by characterization (c) in Theorem 3.2 that C is Schnorr trivial. ∎

One might ask whether one could replace the truth-table reducibility by weak truth-table reducibility or bounded Turing reducibility in the above definition. It will be seen that the Schnorr trivial sets are not closed under either of these two reducibilities, so only strong reducibilities like truth-table reducibility preserve Schnorr triviality. Recall that a weak truth-table (wtt) reduction is a Turing reduction where the use is

bounded by a total recursive function and that a bounded Turing (bT) reduction is a Turing reduction for which there is a constant c such that for any input x, at most c queries are made to the oracle.

Theorem 4.2. *Given any nonrecursive r.e. set A, there is an r.e. set B such that B is not Schnorr trivial, $B \equiv_{wtt} A$ and $B \equiv_{bT} A$. In particular, the Schnorr trivial sets are not closed under $\equiv_{wtt}$ and $\equiv_{bT}$.*

Proof. Let A be given and split the natural numbers recursively into intervals $I_{\langle i,j\rangle}$ of length $2j$. For $x \in A$, let $\Phi^A(x)$ be the stage at which x is enumerated into A; for $x \notin A$, $\Phi^A(x)$ is undefined. Without loss of generality, φ_0 is the function that is 0 everywhere and defined in 0 steps on any input.

Let $\psi(\langle i,j\rangle)$ be $\min(I_{\langle i,j\rangle} - \{\varphi_i(\langle j,k\rangle) : k < |I_{\langle i,j\rangle}| - 1\})$ whenever $j \in A$ and $\varphi_i(\langle j,k\rangle)$ is defined for all $k < |I_{\langle i,j\rangle}| - 1$ within $\Phi^A(j)$ steps. Let B be the range of ψ. B is clearly recursively enumerable.

Now we show that $B \leq_{wtt} A$ and $B \leq_{bT} A$ by showing that $B \leq_{bT} A$ with only one query and that the position of the element queried is given by the recursive function that maps the members of each interval $I_{\langle i,j\rangle}$ to j. The reduction first determines whether j is in A. If not, $B \cap I_{\langle i,j\rangle} = \emptyset$. If so, one can compute $\Phi^A(j)$ and then check to see whether $\psi(\langle i,j\rangle)$ is defined by running each of the corresponding computations $\varphi_i(\langle j,k\rangle)$ for $\Phi^A(j)$ steps. If $\psi(\langle i,j\rangle)$ is defined, then $B \cap I_{\langle i,j\rangle}$ is equal to $\{\psi(\langle i,j\rangle)\}$. Otherwise, let $B \cap I_{\langle i,j\rangle} = \emptyset$. As this reduction is a bounded Turing reduction with only one query and this query is j for any $x \in I_{\langle i,j\rangle}$, the reduction is also a wtt-reduction. This shows that $B \leq_{bT} A$ and that $B \leq_{wtt} A$.

Without loss of generality, assume that $0 \notin A$. If $j \notin A$, then $\psi(\langle 0,j\rangle)$ is undefined and $\min(I_{\langle 0,j\rangle}) \notin B$. If $j \in A$, then $j > 0$ and $0 \notin I_{\langle 0,j\rangle}$, so $\psi(\langle 0,j\rangle) = \min(I_{\langle 0,j\rangle} - \{0\}) = \min(I_{\langle 0,j\rangle})$ as the computations of the numbers $\varphi_0(\langle j,k\rangle)$ involved all terminate within $\Phi^A(j)$ steps. Then $j \in A$ if and only if $\min(I_{\langle 0,j\rangle}) \in B$. Therefore, $A \leq_m B$ and it follows that $A \leq_{wtt} B$ and that $A \leq_{bT} B$.

If B were Schnorr trivial, there would be a total recursive function φ_i such that for all j and x such that $\{x\} = B \cap I_{\langle i,j\rangle}$, x would be in the set $\{\varphi_i(\langle j,k\rangle) : k < |I_{\langle i,j\rangle}| - 1\}$. It now follows from the construction of B that the computation time for the members of the set would be larger than $\Phi^A(j)$ for almost all $j \in A$, as otherwise infinitely many diagonalizations would take place. Hence A would be recursive and we would have a contradiction, so B is not Schnorr trivial.

Proofs of the existence of Schnorr trivial recursively enumerable sets can be found in [5] as well as the next section. Therefore, the Schnorr trivial sets are not closed under $\equiv_{wtt}$ and $\equiv_{bT}$ and not closed downwards under $\leq_{wtt}$ and $\leq_{bT}$. ∎

5 Cohesive Sets

In this section, we show that all maximal sets are Schnorr trivial. This provides a large quantity of natural examples of such sets. Furthermore, we investigate the extent to which this result can be generalized. It turns out that the proof relies heavily on the fact that maximal sets are dense simple, so certain r-maximal sets are not Schnorr trivial. Every maximal set is the complement of a cohesive set, so there are Schnorr trivial cohesive sets. However, this result does not generalize to all cohesive sets and it is shown that only cohesive sets of high Turing degree can be Schnorr trivial. At the end of the section, we give an alternative proof for Franklin's result that there is a Π^0_1-class of Schnorr trivial sets [5].

We recall that a set A is dense simple if it is r.e. and its principal function dominates every recursive function. We further recall that A is hyperhypersimple if there is no disjoint weak array $\{F_n\}_{n\in\mathbb{N}}$ such that for all n, $F_n \cap (\mathbb{N} - A) \neq \emptyset$ and that A is maximal if A is r.e., the complement of A is infinite and there is no r.e. set W such that $W \cap A$ and $W \cap (\mathbb{N} - A)$ are both infinite.

Theorem 5.1. *Every superset of a dense simple set is Schnorr trivial. In particular, maximal and hyperhypersimple sets are Schnorr trivial.*

Proof. Let A be a dense simple set and let B be a superset of A. Let u be a recursive function. Now, for almost all n, $|\{0, 1, 2, \ldots, u(n)\} - A| \leq n$. Hence we can, given n, enumerate A until an s is found such that all but n elements below $u(n)$ are enumerated into A_s and then list the 2^n strings σ of length $u(n) + 1$ which satisfy $\sigma(x) = 1$ for all $x \in \{0, 1, 2, \ldots, u(n)\} \cap A_s$. The string $B(0)B(1)\ldots B(u(n))$ is among these 2^n strings, so the set B is Schnorr trivial by Remark 3.3. The second statement follows from the fact that maximal sets are hyperhypersimple and hyperhypersimple sets are dense simple. ∎

Now we show that this result does not extend to r-maximal sets. Recall that a set A is r-maximal if A is r.e., its complement is infinite and there is no recursive set R such that $R \cap A$ and $R \cap (\mathbb{N} - A)$ are both infinite.

Theorem 5.2. *There is an r-maximal set that is not Schnorr trivial.*

Proof. Stephan [22] proved that there is an r-maximal set A such that, when interpreted as a set of strings, each of its non-members is incompressible for prefix-free Kolmogorov complexity (up to some constant). Now let $f(n)$ be the lexicographically first string in $(\{0, 1\}^n - A) \cup \{1^n\}$. It is clear that $f \leq_{tt} A$ and that $f(n)$ has high Kolmogorov complexity whenever $\{0, 1\}^n \not\subseteq A$. The latter is true for infinitely many

n. It follows that there is no recursive function g such that

$$\forall^\infty n\,[f(n) \in \{g(0), g(1), \ldots, g(2^{n/2})\}]$$

and that therefore A is not Schnorr trivial by Theorem 3.2 (c). ∎

Here, we will say that A has high Turing degree if and only if $A' \geq_T K'$. This is equivalent to the existence of an A-recursive function that dominates every recursive function [13]. Jockusch and Stephan [9] showed that there are cohesive sets that do not have a high Turing degree. Recall that an infinite set A is cohesive if there is no r.e. set W such that both $W \cap A$ and $W \cap (\mathbb{N} - A)$ are infinite. The next result shows that such sets are not Schnorr trivial.

Theorem 5.3. *No cohesive set of nonhigh Turing degree is Schnorr trivial.*

Proof. Let A be a cohesive set of nonhigh Turing degree and let $a_0, a_1, a_2, \ldots$ be a nonrecursive strictly ascending enumeration of A. The function $n \mapsto a_{3^n}$ is A-recursive and, as A is not high, there is a strictly increasing recursive function h such that $h(n) > a_{3^n}$ for infinitely many n. Let $I_n = \{h(n), h(n)+1, \ldots, h(n+1)-1\}$ for all n.

It follows that there are infinitely many n such that $|A \cap I_n| > 2^n$, since otherwise there would be a constant c such that $h(n) \leq a_{2^{n+1}+c}$ for all n, contradicting the fact that $\forall^\infty n\,[2^{n+1} + c < 3^n]$.

Now let $f(n) = \min((A \cap I_n) \cup \{h(n+1)\})$. The function f is truth-table reducible to A. If A were Schnorr trivial, then there would be a recursive function g such that $f(n) \in \{g(2^n), g(2^n+1), \ldots, g(2^{n+1}-1)\}$ for all n. Then we could define the recursive set

$$B = \{g(m) : \exists n\,[2^n \leq m < 2^{n+1} \wedge h(n) \leq g(m) < h(n+1)]\}.$$

As stated before, there are infinitely many n such that $|A \cap I_n| > 2^n$. For these n, the set B contains at least one and at most 2^n of the elements of $A \cap I_n$. Hence there are infinitely many n such that $A \cap I_n \cap B \neq \emptyset$ and $A \cap I_n - B \neq \emptyset$. It follows that $A \cap B$ and $A - B$ are both infinite, contradicting the fact that A is cohesive, so A is not Schnorr trivial. ∎

This result can be generalized to see that the Turing degree of a non-high cohesive set does not contain a Schnorr trivial set. The basic idea of the proof is the same, but the details have to be adjusted.

Corollary 5.4. *If A is cohesive, $A \leq_T B$ and B is not high, then B is not Schnorr trivial.*

Proof. Assume that $A \leq_T B$ and let $u(n)$ be the use function for the computation of a_{3^n} relative to B, where $a_0, a_1, a_2, \ldots$ is a nonrecursive strictly ascending enumeration of A as in the previous theorem. Without loss of generality, u is strictly monotonically increasing and $u(n) > a_{3^n}$ for all n. There is a recursive strictly monotonically increasing function h such that $h(n+1) > u(h(n))$ for infinitely many n. Otherwise, the B-recursive function $\tilde{u}$ inductively defined by $\tilde{u}(0) = 0$ and $\tilde{u}(n+1) = u(\tilde{u}(n))$ would dominate all recursive functions. This is impossible since B is not high.

Now define $f(n)$ to be the maximal element of A computed from B in $h(n+1)$ steps and assume by way of contradiction that B is Schnorr trivial. Note that $f(n) \in \{g(2^n), g(2^n+1), \ldots, g(2^{n+1}-1)\}$ for all n by Remark 3.3. Now let

$$E = \{g(m) : \exists n\, [2^n \leq m < 2^{n+1} \wedge h(n) \leq g(m) < h(n+1)]\}.$$

Suppose that n is one of the infinitely many m such that $u(h(m)) < h(m+1)$. As $h(n) \geq n$, there are more than 3^n elements of A above $h(n)$ which are computed relative to B within $h(n+1)$ steps. One of these elements is $f(n)$, which is in the set E. On the other hand, there are only 2^n elements in E between $h(n)$ and $h(n+1)$, so $A \cap E$ and $A - E$ both have an element between $h(n)$ and $h(n+1)$. It follows that $A \cap E$ and $A - E$ are both infinite, which contradicts the assumption that A is cohesive. Therefore, the assumption that B is Schnorr trivial must be wrong. ∎

This result can be used to give an alternative proof of a result of Franklin [5]. This result shows, together with the basis theorems of Jockusch and Soare [8], that there is a Schnorr trivial set of low Turing degree and one of hyperimmune-free Turing degree. Here we say that a set A has *low Turing degree* if and only if $A' \equiv_T K$ and has *hyperimmune-free Turing degree* if and only if every A-recursive function is majorized by a recursive one [13].

Theorem 5.5 [6]. *There is a Π^0_1-class of Schnorr trivial sets with no recursive members.*

Proof. Let A be a maximal set. By Sacks's Splitting Theorem, there is a partial recursive $\{0,1\}$-valued function ψ with domain A such that the sets $\{x : \psi(x)\downarrow = 0\}$ and $\{x : \psi(x)\downarrow = 1\}$ have incomparable Turing degree. These two sets are recursively inseparable since otherwise one of them would be recursive by the maximality of A. Therefore, ψ has no total recursive extension.

Now consider the class of all sets B whose characteristic function extends ψ. This class is obviously a Π^0_1-class without recursive members. Furthermore, we can see by the density of A that $|\{0,1,2,\ldots,u(n)\} - \mathrm{dom}(\psi)| \leq n$ for all recursive functions u and almost all n. Hence we can compute ψ until all but n places below $u(n)$ are defined and then list the 2^n strings of length $u(n)+1$ which extend this approximation.

This procedure lets us see that each set B whose characteristic function extends ψ is Schnorr trivial by Remark 3.3. ∎

6 A Negative Result

One might ask whether there is also a parallel to Nies's characterization that the sets that are low for Martin-Löf randomness are exactly those sets which are bases for Martin-Löf randomness. The corresponding result for Schnorr randomness turns out to be false, although there is a promising initial proposition.

Proposition 6.1. *Let A and B be sets such that $A \leq_{tt} B$ and B is Schnorr random relative to A. Then A is Schnorr trivial.*

Proof. Let A and B be as desired and let $h(n) = 4^n$. Define u to be the use function of the truth-table reduction from A to B and $q(n)$ to be the fraction of the $2^{u(n)+1}$ strings in $\{0,1\}^{u(n)+1}$ which truth-table compute $A(0)A(1)\dots A(n)$ via the given reduction. Furthermore, let f be an arbitrary recursive, strictly monotonically increasing function.

For each n, we define the martingale mg_n to have the initial value 2^{-n}, increase on any string $\sigma \in \{0,1\}^{u(f(n))+1}$ computing $A(0)A(1)\dots A(f(n))$ via the given truth-table reduction up to the value $2^{-n}/q(f(n))$ and have the value 0 otherwise. The martingale will remain constant after having processed the first $u(f(n))+1$ bits. Then we have two cases.

In the first case, $mg_n(B(0)B(1)\dots B(u(f(n)))) \geq 2^n$ for infinitely many n. Then B cannot be Schnorr random relative to A as the sum $\sum_n mg_n$ is an A-recursive martingale succeeding on B. Since B is Schnorr random relative to A, this does not occur.

In the second case, $mg_n(B(0)B(1)\dots B(u(f(n)))) \leq 2^n$ for almost all n. It follows that $q(f(n)) \geq 4^{-n}$ for almost all n and thus for almost all n, $A(0)A(1)\dots A(f(n))$ is in the list of those 4^n strings which are generated from a set of strings having a share larger than 4^{-n} among all possible values of the string $B(0)B(1)\dots B(u(f(n)))$. Now Remark 3.3 can be used to show that A is Schnorr random. ∎

The second result is a generalization of the result of Calude and Nies [2] that the halting problem is not truth-table reducible to any Martin-Löf random set.

Theorem 6.2. *There is a partial recursive $\{0,1\}$-valued function ψ whose domain is dense simple such that no set A whose characteristic function extends ψ is truth-table reducible to any Schnorr random set.*

Proof. The idea behind this proof is to have intervals I_m of length $2m+1$ that might move from time to time after their initial placement. Each time an interval moves, the characteristic function ψ is defined on the previous values of I_m and no interval will be moved onto these values again. The construction is a priority construction with a bookkeeping set R of pairs $\langle e, m\rangle$ for requirements already satisfied. At the beginning all intervals are unused and R is initialized as the empty set.

Construction. At stage s, we first determine the least m such that I_m requires attention as defined below.

1. I_m requires attention if the interval $I_{m,s}$ is undefined at the current stage.

2. I_m requires attention with respect to $\varphi_e(x)$ if $I_{m,s}$ is defined, $\varphi_{e,s}(x)$ is defined, $e, x \leq m^2$ and $\min(I_{m,s}) \leq \varphi_e(x)$.

3. I_m requires attention with respect to the e-th candidate for a tt-reduction if $I_{m,s}$ exists, this candidate is defined for all elements of $I_{m,s}$ within s steps and $\langle e, m\rangle$ is not yet in R.

Note that there is always some interval I_m that requires attention at stage s. Let m be the minimal index of such an interval and let y be the first number which neither belongs to some defined interval $I_{k,s}$ nor is in the domain of ψ_s.

1. If I_m requires attention because $I_{m,s}$ is undefined, then we define it by letting $I_{m,s+1} = \{y, y+1, y+2, \ldots, y+2m\}$.

2. If I_m requires attention with respect to $\varphi_e(x)$, then we define $\psi_{s+1}(x) = 0$ for all $x \in I_m$ and let $I_{m,s+1} = \{y, y+1, y+2, \ldots, y+2m\}$.

3. If I_m requires attention with respect to the e-th candidate for a truth-table reduction, then we consider all possible $\sigma \in \{0,1\}^{2m+1}$ and select the one for which the e-th candidate has the smallest quantity of inverse images producing it. Define ψ_{s+1} according to σ on $I_{m,s}$ and let $I_{m,s+1} = \{y, y+1, y+2, \ldots, y+2m\}$. Put $\langle e, m\rangle$ into R.

Verification. It is easy to see that every interval I_m requires attention only finitely often. Therefore, I_m has a final value $I_{m,\infty}$ and the domain of ψ is exactly the complement of the union of all intervals $I_{m,\infty}$. It is easy to see that ψ is partial recursive.

Assume now that the e-th candidate is a truth-table reduction. Then each I_m will receive attention with respect to the e-th candidate at some stage, where it has use $u(m)$. The values of ψ on $I_{m,s}$ are such that at most 2^{-2m-1} of the strings of

length $u(m)+1$ are mapped by the e-th candidate to a string which extends ψ on $I_{m,s}$ and we can find these strings effectively. Hence a martingale $\sum_m mg_m$ can be constructed such that mg_m has 2^{-m-1} as its initial value and reaches the value 2^m after querying $u(m)+1$ bits whenever the bits queried produce a string consistent with ψ on $I_{m,s}$. The term mg_m in the sum of the martingale is constant after querying the first $u(m)+1$ bits. It is easy to see that $\sum_m mg_m$ witnesses the statement that no extension of ψ is truth-table reducible via the e-th candidate to a Schnorr random set.

Furthermore, I_m requires attention with respect to some $\varphi_e(x)$ until a stage s is reached such that $\min(I_{m,s}) \geq \varphi_e(x)$ for all $e, x \leq m^2$ where $\varphi_e(x)$ is defined. As the number of the elements in the union $I_{0,\infty} \cup I_{1,\infty} \cup \ldots \cup I_{m,\infty}$ is m^2, it follows that non-element number m^2 of the complement of the domain of ψ is larger than $\varphi_e(x)$ for all $x \leq m^2$. It follows that the domain of φ_e is dense simple. ∎

Remark 6.3. Consider the ψ constructed in Theorem 6.2. We can see that every set whose characteristic function extends ψ is Schnorr random since the domain of ψ is dense simple. Furthermore, ψ has a total extension A of hyperimmune-free Turing degree. Therefore, truth-table Schnorr randomness relative to A coincides with Turing Schnorr randomness relative to A and the various choices given by Proposition 2.1 coincide. This indicates that the negative result of Theorem 6.2 does not depend on the actual choice of the version for truth-table Schnorr randomness relative to A taken and shows that it is not possible to extend the characterization to what might be called a "truth-table basis for Schnorr randomness".

Theorem 6.4. *There is a dense simple set A which is Schnorr trivial and truth-table reducible to a Schnorr random set.*

Proof. Without loss of generality, assume that $\varphi_0(n) = 0$ for all n. Now we can define a recursive one-one function f such that the following two conditions hold.

- The range of f is $\{\langle e, n\rangle : e < n \wedge \varphi_e(n)\downarrow\}$.
- If $f(x) = \langle e, n\rangle$, then $\varphi_e(n) \leq x$.

We will use the fact that φ_0 is always 0 to define $f(x) = \langle 0, n\rangle$ for n whenever no other value can be found. Now let $g(x)$ be the second component of $f(x)$; that is, if $f(x) = \langle e, n\rangle$, then $g(x) = n$. Let

$$A = \{x : \exists y > x\, [g(y) = g(x)]\}.$$

The set A is recursively enumerable. Furthermore, A is dense simple, since for every total function φ_e and every $n > e$, if $f(x) = n$ and $x \notin A$, then $x \geq \varphi_e(n)$.

Now let $I_0, I_1, I_2, \ldots$ be a recursive partition of $\mathbb{N}$ into intervals such that $|I_m| =$

$g(m)$ for all m. Let mg be the universal martingale that identifies sets that are not Martin-Löf random. Note that mg is approximable from below but not recursive, since its initial value is a left-r.e. set number between 0 and 1.

Now one defines a set R inductively as follows. On an interval I_m with $m \in A$, R is chosen such that mg grows at most by a factor $1/(1-2^{-m})$ and is not 0 on all elements of I_m. No element of an interval I_m with $m \notin A$ is put into R and mg can grow by a factor of 2^m on this interval. It is clear that $A \leq_{tt} R$, since $m \in A$ if and only if $I_m \cap R \neq \emptyset$.

Assume now for a contradiction that R is not Schnorr random. Let r be the value of the factor by which mg can grow on intervals I_m with $m \in A$; that is,

$$r = \prod_{m>0} (1-2^{-m})^{-m}.$$

Note that this product is convergent. As R is assumed not to be Schnorr random, there is a recursive function f such that $mg(R(0)R(1)\ldots R(f(n))) > n$ for infinitely many n. Choose e such that $\varphi_e(m) = f(0)+f(1)+\ldots+f(\lceil r \cdot 2^{(m+1)(m+2)} \rceil)+m$ for all m. By construction there are, for $m > e$, only m intervals I_n such that $I_n \cap R = \emptyset$ below $\varphi_e(m)$ and these intervals have lengths $1, 2, 3, \ldots, m$, respectively. On these m intervals, mg may increase its value by a factor of $2^{m(m+1)}$. Outside the intervals I_n such that $I_n \cap R = \emptyset$, the value of mg grows by at most a factor of r. As a consequence, for all $m > e$ and all k such that $r \cdot 2^{m(m+1)} < k \leq r \cdot 2^{(m+1)(m+2)}$,

$$mg(R(0)R(1)\ldots R(f(k))) \leq r \cdot 2^{m(m+1)} < k$$

and so the assumption that mg grows on the first $f(k)+1$ bits to the value k for infinitely many k turns out to be false. Therefore, R is indeed Schnorr random and A is truth-table reducible to a Schnorr random set. ∎

Question 6.5. Do these Schnorr trivial sets truth-table reducible to Schnorr random sets form a truth-table ideal? In other words, if A and B are both Schnorr trivial sets that are truth-table reducible to a Schnorr random set, does the same hold for $A \oplus B$?

One might try to prove a similar statement by considering alternative reducibilities, in particular, those for which the Schnorr trivial sets form the bottom degree. Besides the reducibility $\leq_{Sch}$ considered by Downey, Griffiths and LaForte [3], one can also consider the following "Schnorr-style reducibility" $\leq_{snr}$.

Definition 6.6. We say that $A \leq_{snr} B$ if and only if

$$\exists \text{ recursive } h\, \forall f \leq_{tt} A\, \exists g \leq_{tt} B\, \forall n\, \exists m \leq h(n)\, [f(n) = g(m)].$$

We can see by Theorem 3.2 (c) that $A \leq_{snr} \emptyset$ if and only if A is Schnorr trivial. This result can now be extended to obtain a theorem similar to those involving bases for randomness, although the reducibility is not the expected one. Note that for every fixed set B, the class $\{A : A \leq_{snr} B\}$ is a truth-table ideal.

Theorem 6.7. *A is Schnorr trivial if and only if there is a set B such that B is truth-table Schnorr random relative to A and $A \leq_{snr} B$.*

Proof. If A is Schnorr trivial, then A is snr-reducible to every set, so we will only prove the other direction. Assume that $A \leq_{snr} B$ and that B is truth-table Schnorr random relative to A. We will show that A is Schnorr trivial by arguments adapted from the proof of Proposition 6.1.

Let the recursive bound h from the snr-reduction be given. Consider any recursive and strictly monotonically increasing function f. We will list $4^n \cdot h(n)$ strings, including $A(0)A(1)\dots A(f(n))$, for every n.

Let u be the use function for the truth-table reduction that computes up to $h(n)$ strings, including $A(0)A(1)\dots A(f(n))$, for input n. Let $q(n) = r(n)/2^{u(n)+1}$, where $r(n)$ is the number of strings in $\{0,1\}^{u(n)+1}$ such that $A(0)A(1)\dots A(f(n))$ is among the 2^n candidates produced by the given truth-table reduction.

For each n, consider the martingale mg_n that has initial value 2^{-n} and goes up to the value $2^{-n}/q(n)$ on any string $\sigma \in \{0,1\}^{u(f(n))+1}$ that produces $h(n)$ candidates, including $A(0)A(1)\dots A(f(n))$, via the given truth-table reduction. The martingale remains constant after having queried the first $u(f(n))+1$ bits. There are two cases.

In the first case, $mg_n(B(0)B(1)\dots B(u(f(n)))) \geq 2^n$ for infinitely many n. Then B cannot be Schnorr random relative to A, since the sum $\sum_n mg_n$ is an A-recursive martingale that succeeds on B. As B is Schnorr random relative to A, this case will not occur.

In the second case, $mg_n(B(0)B(1)\dots B(u(f(n)))) \leq 2^n$ for almost all n. It follows that $q(n) \geq 4^{-n}$ for almost all n. As there are 2^n candidates generated by the truth-table reduction, we can conclude that there are at most $4^n \cdot h(n)$ strings $\sigma \in \{0,1\}^{f(n)+1}$ such that at least $2^{u(n)+1-2n}$ strings $\tau \in \{0,1\}^{u(n)+1}$ produce a list of $h(n)$ strings including σ under the given truth-table reduction. These $4^n \cdot h(n)$ strings can be listed and $A(0)A(1)\dots A(f(n))$ will be among them for almost all n. This shows that A is Schnorr trivial by Remark 3.3. ∎

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