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Confident and Consistent Partial Learning of Recursive Functions

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T e c h n i c a l   R e p o r t

Foreword

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Confident and Consistent Partial Learning of Recursive Functions

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Abstract. Partial learning is a criterion where the learner infinitely often outputs one correct conjecture while every other hypothesis is issued only finitely often. This paper addresses two variants of partial learning in the setting of inductive inference of functions: first, confident partial learning requires that the learner also on those functions which it does not learn, singles out exactly one hypothesis which is output infinitely often; second, essentially class consistent partial learning is partial learning with the additional constraint that on the functions to be learnt, almost all hypotheses issued are consistent with all the data seen so far. The results of the present work are that confident partial learning is more general than explanatory learning, incomparable with behaviourally correct learning and closed under union; essentially class consistent partial learning is more general than behaviourally correct learning and incomparable with confident partial learning. Furthermore, it is investigated which oracles permit to learn all recursive functions under these criteria: for confident partial learning, some non-high oracles are omniscient; for essentially class consistent partial learning, all PA-complete and all oracles of hyperimmune Turing degree are omniscient.

1 Introduction

Gold [8] initiated the study of inductive inference, which investigates various forms of learning recursive functions and r.e. sets in the limit. Gold originally considered recursive learners which receive piecewise information about the graph of an unknown recursive function, presented in the natural ordering of the input values, while they conjecture a sequence of hypotheses which syntactically converges to a correct conjecture. Osherson, Stob and Weinstein [16] generalised Gold's paradigm to *partial learning* by weakening the convergence requirement in such a way that one correct hypothesis is required to be conjectured infinitely often while every other hypothesis is conjectured only finitely often.

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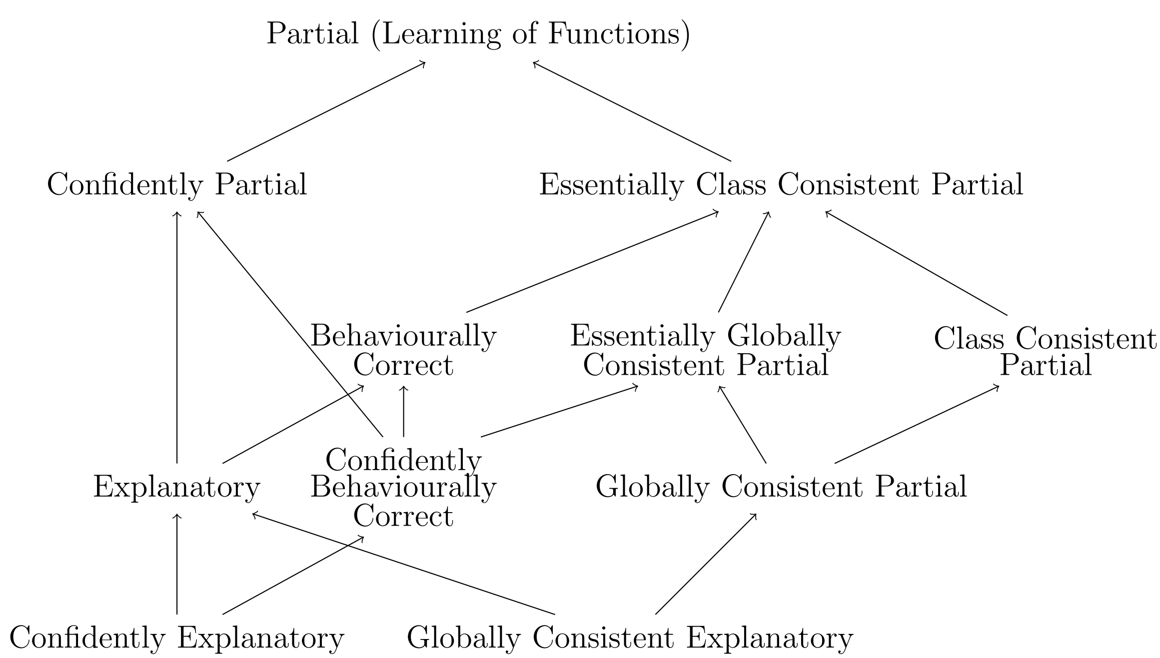
On one hand, many natural examples of classes of recursive functions fail to be identifiable in the limit by any recursive learner, even in the broadest sense of semantic convergence [1]; this deficiency has motivated alternative approaches to learnability in the inductive inference such as the above mentioned one of partial learning. Feldman [5], for example, showed that a decidable rewriting system (drs) is always learnable from positive information sequences in a certain restricted sense. When introducing their criterion, Osherson, Stob and Weinstein [16] discovered that the whole class REC of recursive functions is partially learnable and that partial learnability is much more general even than behaviourally correct learnability. Subsequently, researchers thought that partial learning is too general and studied what happens when partial learning is combined with more restrictive constraints, most notably consistency which was introduced by Bārzdiņš [1] and which means that each hypotheses e for some data $f(0), f(1), \ldots, f(n)$ satisfies that $\varphi_e(m)$ is defined and equal to m for all $m \leq n$. Indeed, consistent partial learners can easily be shown to fail learning the class of all recursive functions. Wiehagen and Zeugmann [19] and later Grieser [9] and Jain and Stephan [12] studied consistent learning and partial consistent learning. Other constraints of partial learning were neglected, mostly as the corresponding notions coincided with partial learning itself.

The present work wants to fill this gap; as a start, the notion of confident partial learning is brought over to function learning from the original setting of language learning for which it was introduced by Gao, Stephan, Wu and Yamamoto [7]. In addition the present work introduces the notions of essentially class consistent and essentially globally consistent partial learning; these learning notions align as follows with other notions of inductive inference:



In the following, the concepts and results are explained in more detail. Confidence in partial learning enforces that the learner must issue exactly one hypothesis infinitely often when reading the data of one given object, even if this object does not belong to the target class. In the case of language learning, the notion turned out to be restrictive [7]: even the class of all cofinite sets is not confidently partially learnable.

On the other hand, confident partial learning has some regularity properties. In the here investigated case of function learning, one can show that the union of confidently partially learnable classes is confidently partially learnable (this is parallel to the corresponding result for confidently explanatory learning of classes of functions); furthermore, this notion is more general than Gold's original notion of explanatory learning [3, 8] and incomparable to the more general notion of behaviourally correct learning [1]. Confidence, though restrictive, is nevertheless a desirable quality of a learner as the learner tries always to come up with a hypothesis, even in the case that the data is arbitrary. This property permits to prove some desirable aspects of confidently learnable classes, for example, that the union of two confidently learnable classes is again confidently learnable.

Consistency, whilst a fairly stringent learning constraint, may be quite a desirable quality of learners, especially when the inductive inference paradigm is viewed as a model for scientific discovery. It is conceivable that a scientific theory with any epistemic value must be developed in accordance with empirical data, and, while allowing for a certain margin of error due to experimental inaccuracies, should possess a set of potential falsifiers that determine the consistency or non-consistency of its fundamental assumptions under the conditions of a controlled experiment [14]. Briefly, the falsificationist methodological rule expounded by Popper [17] states that a scientific theory is to be rejected if it is inconsistent with some basic statement unanimously accepted by the scientific community. In view of this benchmark by which science progresses, one may argue that consistency with empirical data is an essential characteristic of the hypotheses issued by scientists modelled as recursive learners.

Jain and Stephan [12] showed that the class REC of all recursive functions can be consistently partially learnt relative to an oracle A iff A has hyperimmune degree. In the present paper, we show that by weakening this learning constraint to *essential consistency*, under which a recursive learner is only required to be consistent on cofinitely many segments of a sequence input, REC can be partially inferred relative to any PA-complete oracle. Thus, by the result of Jockusch and Soare [13] that there are hyperimmune-free PA-complete sets, one can conclude that there is a strictly larger family of oracles relative to which REC is essentially class consistently partially learnable. The main result for this notion is that it is still more general than behaviourally correct learning; this is a surpising result as usually the generalisations of behaviourally correct learning are either obtained by varying the concept of semantic convergence (for example, by augmenting it with errors) or by taking a notion which is already learning the full class REC. Further results on essentially class consistent learning in the present work are that this notion is neither closed under union nor comparable to confident partial learning.

2 Notation

The notation and terminology from recursion theory adopted in this paper follows the book of Rogers [18] in the main. Background on inductive inference can be found in [11]. $\mathbb{N}$ denotes the set of natural numbers. Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ denote a fixed acceptable numbering of all partial-recursive functions. Given a set S, $\overline{S}$ denotes the complement of S, and S^* denotes the set of all finite sequences whose elements are drawn from S. Let $W_0, W_1, W_2, \ldots$ be a universal numbering of all r.e. sets, where W_e is the domain of φ_e. $\langle x, y\rangle$ denotes Cantor's pairing function, given by $\langle x, y\rangle = \frac{1}{2}(x+y)(x+y+1)+y$. $W_{e,s}$ is an approximation to W_e; without loss of generality, $W_{e,s} \subseteq \{0, 1, \ldots, s\} \cap W_{e,s+1}$ and the set $\{\langle e, x, s\rangle : x \in W_{e,s}\}$ is primitive recursive. $\varphi_e(x)\uparrow$ means that $\varphi_e(x)$ remains undefined; $\varphi_{e,s}(x)\downarrow$ means that $\varphi_e(x)$ is defined, and that the computation of $\varphi_e(x)$ halts within s steps. Turing reducibility is denoted by $\leq_T$; $A \leq_T B$ holds iff A can be computed via a machine which knows B, that is, for any given x, it gives information on whether or not x belongs to B. $A \equiv_T B$ means that $A \leq_T B$ and $B \leq_T A$ both hold, and $\{A : A \equiv_T B\}$ is called the Turing degree of B. The class of all recursive functions is denoted by REC; the class of all $\{0,1\}$-valued recursive functions is denoted by $REC_{0,1}$. For any two partial-recursive functions f and g, $f =^* g$ denotes that for cofinitely many x, either $f(x)$ and $g(x)$ are both undefined or $f(x)\downarrow = g(x)\downarrow$; for any number a, $f =^a g$ denotes that for all but at most a values of x, either $f(x)$ and $g(x)$ are both undefined or $f(x)\downarrow = g(x)\downarrow$. The symbol $\mathbb{K}$ denotes the halting problem. The jump of a set A is denoted by A' and denotes the relativised halting problem $A' = \{e : \varphi_e^A(e)\downarrow\}$. For any two sets A and B, $A \oplus B = \{2x : x \in A\} \cup \{2y+1 : y \in B\}$. Let pad be a two-place recursive function such that $\varphi_{pad(e,d)} = \varphi_e$ and $pad(e,d) \neq pad(e', d')$ if $(e,d) \neq (e', d')$ for all numbers e, d, e', d'.

For any $\sigma, \tau \in (\mathbb{N} \cup \{\#\})^*$, $\sigma \preceq \tau$ iff $\sigma = \tau$ or τ is an extension of σ, $\sigma \prec \tau$ iff σ is a proper prefix of τ, and $\sigma(n)$ denotes the element in the nth position of σ, starting from $n = 0$. $|\sigma|$ is the length of σ. $\sigma(n)\downarrow$ means that $\sigma(n)$ is defined, that is, $n < |\sigma|$. The domain of σ, denoted by $\mathrm{dom}(\sigma)$, is the set of values of n for which $\sigma(n)\downarrow$. Given a number a and some fixed $n \geq 1$, denote by a^n the finite sequence $a \ldots a$, where a occurs n times. a^0 denotes the empty string. The concatenation of two strings σ and τ shall be denoted by $\sigma\tau$ and occasionally by $\sigma \circ \tau$. For a total function f, $f[k]$ denotes the sequence $f(0)f(1)\ldots f(k)$.

Jockusch and Soare [13], as well as Hanf [10], studied the Turing degrees which permit to compute an infinite branch in every infinite recursive binary tree. They showed that these Turing degrees coincide with those which permit to compute a complete extension of the first order version of Peano Arithmetic; thus such degrees are called *PA-complete*. The following definition provides an easy way to formalise PA-completeness and other basic recursion-theoretic notations like low, high, high_2 and the various levels of genericity.

Definition 1. A set A is *PA-complete* iff, given any partial-recursive and $\{0,1\}$-valued function ψ, one can compute relative to A a total extension Ψ of ψ.

Let $\mathbb{K}$ denote the halting problem and let A' denote the halting problem relative to A: $(e,x) \in A' \Leftrightarrow \varphi_e^A(x)$ is defined. Here, for any function g which might be computed relative to some oracle, the notation g^A means that the oracle used for the computation is A.

A set A is *low* iff its jump is Turing equivalent to the halting problem: $A' \equiv_T \mathbb{K}$. A set A is *high* iff its jump is Turing above the jump of the halting problem: $A' \geq_T \mathbb{K}'$.

A set A is *high*$_2$ iff its double jump is Turing above the double jump of the halting problem: $A'' \geq_T \mathbb{K}''$. For any m, a set A is *m-generic* iff for every Σ^0_m set $W \subseteq \{0,1\}^*$ there is an n such that either $A(0) \circ A(1) \circ \ldots \circ A(n) \in W$ or no extension of $A(0) \circ A(1) \circ \ldots \circ A(n)$ belongs to W.

In the above, the common convention is used that if $x \in A$ then $A(x) = 1$ else $A(x) = 0$.

Note that the low and high sets have an alternate characterisation. An oracle A is high iff there is an A-recursive function f which dominates all recursive functions g, that is, which satisfies $\forall$ recursive $g\,\exists n\,\forall m > n\,[f(m) > g(m)]$. An oracle A is low iff $A \leq_T \mathbb{K}$ and there is a $\mathbb{K}$-recursive function which dominates all A-recursive functions.

3 Learnability

Let $\mathcal{C}$ be a class of recursive functions. Throughout this paper, the mode of data presentation is that of an infinite sequence whose ith term is $f(i)$, where f is some total function. The main learning criteria studied in this paper are *partial learning*, *explanatory learning* and *behaviourally correct learning*. M is a recursive function mapping $\mathbb{N}^*$ into $\mathbb{N}$.

i. Osherson, Stob and Weinstein [16] defined that M *partially* ($Part$) learns $\mathcal{C}$ iff, for each f in $\mathcal{C}$, there is exactly one index e such that $M(f[k]) = e$ for infinitely many k; this index e also satisfies $f = \varphi_e$.
ii. Gold [8] defined that M *explanatorily* (Ex) learns $\mathcal{C}$ iff, for each f in $\mathcal{C}$, there is a number n for which $f = \varphi_{M(f[n])}$ and, for any $j \geq n$, $M(f[j]) = M(f[n])$.
iii. Bārzdiņš [1] defined that M *behaviourally correctly* (BC) learns $\mathcal{C}$ iff, for each f in $\mathcal{C}$, there is a number n for which $f = \varphi_{M(f[j])}$ whenever $j \geq n$.

The next two definitions impose additional constraints on the learner.

Definition 2. i. Gao, Stephan, Wu and Yamamoto [7] defined that a recursive learner M *confidently partially* learns $\mathcal{C}$ iff it partially learns $\mathcal{C}$ and outputs on every infinite sequence exactly one index infinitely often.
ii. A recursive learner M is said to *essentially class consistently partially* learn $\mathcal{C}$ iff it partially learns $\mathcal{C}$ and, for each f in $\mathcal{C}$, $\varphi_{M(f[n])}(m)\downarrow = f(m)$ holds whenever $m \leq n$ for cofinitely many n.
iii. A recursive learner M is said to *essentially globally consistently partially* learn $\mathcal{C}$ iff it partially learns $\mathcal{C}$ and, for each function f, $\varphi_{M(f[n])}(m)\downarrow = f(m)$ holds whenever $m \leq n$ for cofinitely many n.

4 Confident partial learning

The first learning constraint proposed here as a means of sharpening partial learnability is that of *confidence*. Osherson, Stob and Weinstein [16] introduced confidence for explanatory and other learning notions. They defined that a confident learner provides on each input function a hypothesis with respect to the given learning criterion, and this hypothesis has to be correct on all functions in the class to be learnt. For example, in the case of explanatory learning, this means that the learner converges on every function syntactically to some hypothesis, even if the function is not recursive and therefore cannot have an index at all. Similarly, in the case of behaviourally correct learning, the convergence is semantic and so for every f, the corresponding sequence $e_0, e_1, \ldots$ of hypotheses of the learner satisfies that there is a partial-recursive function ψ with $\varphi_{e_n} = \psi$ for almost all n. The constraint of confidence in partial learning is, correspondingly, that on every f the learner outputs exactly one index infinitely often. It is known that confidence is a real restriction for explanatory and behaviourally correct learning compared to the non-confident versions of the respective learning criteria. The following result shows that confidence is also restrictive for partial learning; there is, in fact, a class which is behaviourally correctly learnable but not confidently partially learnable.

Theorem 3. *There is a behaviourally correctly learnable class of recursive functions which is not confidently partially learnable.*

Proof. Given a recursive function g to be specified below, consider the class

$$\begin{aligned}\mathcal{C} = \{f : f &\text{ is recursive and } \{0,1\}\text{-valued and there is an } e \text{ such that} \\ &f(0) = 0 \wedge f(1) = 0 \wedge \ldots \wedge f(e-1) = 0 \wedge f(e) = 1 \\ &\wedge\ \varphi_{g(e)}(x) \text{ is defined for almost all } x \\ &\wedge\ \varphi_{g(e)}(x) \text{ is always either undefined or equal to } f(x)\}.\end{aligned}$$

A behaviourally correct learner M outputs a default index 0 until it witnesses the first number e with $f(e) = 1$; subsequently, on the input $\sigma = 0^e \circ 1 \circ f(e+1) \circ \ldots \circ f(e+k)$, it conjectures an index i with $\varphi_i(x) = \sigma(x)$ if $x < |\sigma|$ and $\varphi_i(x) = \varphi_{g(e)}(x)$ otherwise.

The function g will be chosen such that the existence of an A-recursive confident partial learner N will give an A''-recursive procedure to decide whether $d \in \{e : W_e \text{ is cofinite}\}$ for any given d; this will then enforce that such an A must be high$_2$. More precisely, let g be a recursive function for which $\varphi_{g(d)}$ is defined in stages as follows:

- Set $\varphi_{g(d),0}(x) = 0$ for $x < d$, $\varphi_{g(d),0}(d) = 1$ and $\varphi_{g(d),0}(x)\uparrow$ for all $x > d$. Initialise the markers $a_0, a_1, a_2, \ldots$ by setting $a_{i,0} = \langle i, 0\rangle + d + 1$ for $i \in \mathbb{N}$.
- At stage $t+1$, choose i, j such that $t = \langle i, j\rangle$ and consider the marker $a_{i,t}$ which sits on a position $a_{i,t} = \langle i, j'\rangle + d + 1$ with $j' \leq j$.
- Now do the first of the following cases which applies:
  - If $\varphi_{i,t}(a_{i,t})$ is defined and in $\{0,1\}$ then let $\varphi_{g(d)}(a_{i,t}) = 1 - \varphi_{i,t}(a_{i,t})$ and $a_{i,t+1} = 1 + d + \langle i, j+1\rangle$;
  - If $|\{0,1,\ldots,j'\} - W_{d,t}| \leq i$ then let $\varphi_{g(d)}(a_{i,t}) = 0$ and $a_{i,t+1} = 1 + d + \langle i, j+1\rangle$;
  - If $j' < j$ then let $\varphi_{g(d)}(\langle i, j\rangle + d + 1) = 0$ and $a_{i,t+1} = a_{i,t}$;
  - Otherwise $\varphi_{d,t+1} = \varphi_{d,t}$ and $a_{i,t+1} = a_{i,t}$.
- For all $i' \neq i$, let $a_{i',t+1} = a_{i',t}$.

It shall be shown that the partial-recursive function $\varphi_{g(d)}$ as defined above possesses the following properties:

1. $\varphi_{g(d)}(\langle i,j\rangle + d + 1)$ is undefined only if $a_{i,t} = \langle i,j\rangle + d + 1$ for all $t \geq \langle i,j\rangle + 1$.
2. If W_d is cofinite then every marker a_i with $i > |\overline{W_d}|$ moves infinitely often and $W_{g(d)}$ is also cofinite.
3. If W_d is coinfinite, then every i with φ_i being total satisfies that there is a j with $\varphi_i(\langle i,j\rangle + d + 1) \neq \varphi_{g(d)}(\langle i,j\rangle + d + 1)$ and $W_{g(d)}$ is coinfinite.

Item 1 follows from the construction and the way the markers move.

Item 2 follows from the fact that if $i > |\overline{W_d}|$ and $a_{i,t} = \langle i,j\rangle + d + 1$ then there is some $j' > t$ and $t' = \langle i,j'\rangle$ such that in stage $t'+1$ it holds that $|\{0,1,\ldots,j\} - W_{d,t'}| \leq i$ and so the marker leaves position $\langle i,j\rangle + d + 1$ latest in stage $t'+1$ due to the first or second of the conditions on its movement.

Item 3 follows from the fact that if W_d is coinfinite then $\{0,1,\ldots,j'\} - W_d$ contains for almost all j' more than i elements and therefore either the marker a_i will eventually move due to the first condition which makes φ_i and $\varphi_{g(d)}$ explictly to be different or the marker a_i will eventually reach a position from which it abstains to move and where φ_i is undefined or takes a value in $\{2,3,\ldots\}$. Thus no total function φ_i extends $\varphi_{g(d)}$ and the domain of $\varphi_{g(d)}$ must be non-recursive and coinfinite.

Now let A be any oracle such that there is an A-recursive confident partial learner N for the class. By the Low Basis Theorem of Jockusch and Soare relativised to A there is an oracle $B \geq_T A$ which is PA-complete and satisfies $B' \equiv_T A'$. Hence, as B is PA-complete, there is a uniformly B-recursive $\{0,1\}$-valued extension G_d^B of the partial-recursive function $\varphi_{g(d)}$. There is a further recursive function h for which $W_{h(d,e)}^B = \{n :\ N$ outputs e at least n times when trying to learn $G_d^B\}$. Owing to the confidence of N, one can determine by means of the oracle B'' the unique e such that $W_{h(d,e)}^B$ is infinite.

If W_d is cofinite, then, as was shown above, the domain of $\varphi_{g(d)}$ is also cofinite and G_d^B is a total recursive extension of $\varphi_{g(d)}$. Therefore N learns the recursive function G_d^B and consequently $\varphi_e(x) = G_e^B(x)$ for all x.

However, if W_d is coinfinite, it follows from the construction of $\varphi_{g(d)}$ that there is no total recursive function extending $\varphi_{g(d)}$, giving that $\varphi_e \neq G_d^B$. More specifically, there is an x such that either $\varphi_e(x)\uparrow$ or $\varphi_e(x)\downarrow\neq G_d^B(x)$.

Hence W_d is cofinite iff for all x, $\varphi_e(x)\downarrow= G_d^B(x)$. As this condition may be checked using the oracle B'', as $B' \equiv_T A'$ and as $B'' \equiv_T A''$, $\mathbb{K}'' \equiv_T \{d : W_d \text{ is cofinite}\} \leq_T A''$ and A is a high$_2$ oracle, in particular, A is not recursive. Therefore the class $\mathcal{C}$ cannot be confidently partially learnt in the unrelativised world. ∎

The following theorem formulates a criterion that may appear at first sight to be less stringent than confident partial learnability, but is in fact equivalent to it. The proof illustrates a padding technique, dependent on the underlying hypothesis space of the learner, that is often applied throughout this work to construct confident partial learners.

Theorem 4. *A class $\mathcal{C}$ of recursive functions is confidently partially learnable iff there is a recursive learner M such that*

- *M outputs on each infinite sequence exactly one index infinitely often;*
- *if M outputs d infinitely often on the sequence for a recursive function f in $\mathcal{C}$, then there is an index e of f with $e \leq d$.*

Proof. Suppose that there is a recursive learner M of $\mathcal{C}$ which satisfies the learning criteria laid out in the statement of the theorem. One may define a learner N which confidently partially learns $\mathcal{C}$ as follows: on the input sequence $f(0) \circ f(1) \circ \ldots \circ f(n) \circ \ldots$, N outputs $pad(e,d)$ at least n times if

- N has output d many indices of the form $pad(e',d')$ with $e' < e$ among its first n hypotheses, and
- either $\varphi_e(x) \downarrow = f(x)$ for all $x < n$ or M has output e at least n times.

For the verification, assume that e is the least index such that either M outputs e infinitely often or $f = \varphi_e$. Consider $e' < e$ and the least $d_{e'}$ such that $\varphi_{e'}(x)$ differs from $f(x)$ for some $x < d_{e'}$ and M does not output e' $d_{e'}$ times. Then N will also at most $d_{e'}$ times output an index of the form $pad(e',d')$. Furthermore, let d be the number of times an index of the form $pad(e',d')$ with $e' < e \wedge d' \in \mathbb{N}$ is output by N. Then N will output $pad(e,d)$ infinitely often and that is the only index output infinitely often by N when processing f.

For the converse direction, any given confident partial learner of $\mathcal{C}$ clearly satisfies the conditions on M given in the statement of this theorem. ∎

Definition 5. A class is Ex^1-learnable iff there is a learner M which converges on the sequence $f(0)f(1)f(2)\ldots$ for any function f in this class to an index e such that, for all but at most one x, $\varphi_e(x)\downarrow = f(x)$.

Theorem 6. *Every Ex^1-learnable class is confidently partially learnable.*

Proof. Assume that M is an Ex^1-learner for a class $\mathcal{C}$, where, without loss of generality, $M(\sigma \circ \tau) \geq M(\sigma)$ for all σ, τ. Furthermore let *patch* be a recursive function with $\varphi_{patch(e,x,y)}(x) = y$ and $\varphi_{patch(e,x,y)}(z) = \varphi_e(z)$ for all $z \neq x$; without loss of generality, *patch* is a one-one function. Now one constructs a new confident partial learner N as follows:

- N outputs $patch(e, x, f(x))$ at least n times if M has at least n times output the index e and $\varphi_e(x)$ does not output $f(x)$ within n steps while $\varphi_e(z)$ is defined and equal to $f(z)$ for all $z < x$;
- N outputs $patch(e, 0, f(0))$ at least n times if M has at least n times output the index e and $\varphi_e(z)$ is defined and equal to $f(z)$ for all $z < n$;
- N outputs $patch(0, 0, 0)$ at least n times if M makes on f at least n mind changes.

One can see the following: If M diverges on f then N outputs the hypothesis $patch(0, 0, 0)$ infinitely often and all other hypotheses only finitely often. If M converges to a correct index

e on f then N outputs $patch(e, 0, f(0))$ infinitely often and all other indices only finitely often. If M converges on f to an index e which differs on at least one value from f by either being undefined or being wrong then N outputs $patch(e, x, f(x))$ infinitely often where x is the least number where φ_e is either undefined or different from f. This shows that N is a confident partial learner for the given class. ∎

Remark 7. The preceding result is a generalisation of the statement that every explanatorily learnable class is confidently learnable. Indeed, note that the class $\mathcal{C} = \{f : f(0)$ is an index for f which is correct at all but at most one inputs$\}$ is Ex^1-learnable and behaviourally correctly learnable but not explanatorily learnable. One could easily generalise the result such that one shows that every Ex^a-learnable class where the learner converges to an index with at most a errors is confidently partially learnable where $a \in \{0, 1, 2, \ldots\}$ is a fixed constant. The more general criterion of Ex^*-learnable classes is not covered by confident partial learning as the class from Theorem 3 shows. Case and Smith [4] studied the learnability of these type of self-describing function classes extensively.

One might also ask, how confident partial learning is related to confident behaviourally correct learning which is a more restrictive notion than normal behaviourally correct learning. The definition of this learning notion is the following.

Definition 8. A class of recursive functions $\mathcal{C}$ is confidently behaviourally correctly learnable iff there is a learner M which behaviourally correctly learns every $f \in \mathcal{C}$ and in addition has the property that it on every function f, even on non-recursive ones, outputs a sequence of conjectures $e_0, e_1, e_2, \ldots$ which stabilises semantically, that is, which satisfies $\varphi_{e_n} = \varphi_{e_{n+1}}$ for almost all n.

Theorem 9. *If $\mathcal{C}$ is confidently behaviourally correctly learnable then $\mathcal{C}$ is confidently partially learnable.*

Proof. Assume that M is a behaviourally correct learner for $\mathcal{C}$ and that f is any function. Now let $e_0, e_1, e_2, \ldots$ be the sequence of hypotheses issued by M on f; without loss of generality one can assume padding so that $e_0 < e_1 < \ldots$ which is important for having a simpler algorithm for N. N now mainly copies and repeats hypotheses of M. Each hypothesis d_n of N is equal to some e_m. The basic idea of N is to do a mix of repeating and cancelling indices; indices which are cancelled will never be repeated again while all others might, but do not need to, qualify for future repeats. In step n, the learner N selects d_n as follows:

- Say i qualifies on level k at step n if i has not yet been cancelled at previous steps and for all $x < k$, either it holds that for all $j \in \{i, i+1, \ldots, i+k\}$ the values $\varphi_{e_j,n}(x)$ are defined and equal or it holds that for all $j \in \{i, i+1, \ldots, i+k\}$ the values $\varphi_{e_j,n}(x)$ are undefined;
- For each i let $k_i = |\{m < n : d_m = e_i\}|$ and select the least i which qualifies on level k_i;
- For the i selected, let $d_n = e_i$ and cancel $e_{i+1}, e_{i+2}, \ldots, e_{i+n}$.

Note that every i for which there is a $j > i$ with $\varphi_{e_j} \neq \varphi_{e_i}$ will qualify only finitely often. As the sequence $e_0, e_1, e_2, \ldots$ converges semantically to a partial function, there are only finitely many $j > i$ with this property. Therefore there is a first step n from which onwards only i qualify which satisfy $\varphi_{e_j} = \varphi_{e_i}$ for all $j > i$. Now let i be the least number such that $\varphi_{e_j} = \varphi_{e_i}$ for all $j > i$ and i has not yet been cancelled before step n. Then from step n onwards, no $i' < i$ will be selected and hence i will never be cancelled; furthermore, there will be infinitely many steps $m \geq n$ where $d_m = e_i$. Furthermore, all $j > i$ will be cancelled eventually. Hence the index e_i is the unique index which occurs infinitely often in the sequence $d_0, d_1, d_2, \ldots$ of hypotheses of N. If M behaviourally correctly learns f then $\varphi_{e_j} = f$ for almost all j and thus $\varphi_{e_i} = f$. Thus N is both, a confident behaviourally correct and a confident partial learner for $\mathcal{C}$. ∎

Remark 10. The notion of confident behaviourally correct learning is incomparable to Ex^a-learning. On one hand, the class of the almost everywhere constant functions is explanatorily learnable but not confidently behaviourally correctly learnable. On the other hand, the class $\{f : f =^{f(0)} \varphi_{f(0)}\}$ is confidently behaviourally correctly learnable but not Ex^a-learnable for any $a \in \mathbb{N}$. The cylindrification of this class would result in a confidently behaviourally correctly learnable class which is not Ex^*-learnable.

It is quite a curious feature of confident learning under various success criteria that it is closed under finite unions. In particular, it is known that the union of finitely many confidently vacillatorily learnable classes is also confidently vacillatorily learnable; the analogous result for confident behaviourally correct learning also holds true. The next theorem states that this property of confident learning even extends to partial learnability. That is to say, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are confidently partially learnable classes of recursive functions, then $\mathcal{C}_1 \cup \mathcal{C}_2$ is also confidently partially learnable.

Theorem 11. *Confident partial learning is closed under finite unions; that is, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are confidently partially learnable classes, then $\mathcal{C}_1 \cup \mathcal{C}_2$ is confidently partially learnable.*

Proof. Let M and N be confident partial learners of the classes $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively. Now using Theorem 4, one can construct a new learner R which outputs $\langle i, j\rangle$ at least n times iff M outputs i and N outputs j at least n times. Now assume that f is a function which is learnt by at least one of the machines M and N. So let i and j be the two indices which M and N, respectively, output infinitely often when processing f and let e be the least index of f. Note that either $i \geq e \vee j \geq e$. Now R outputs $\langle i, j\rangle$ infinitely often. As Cantor's pairing function is monotone in both parameters, $\langle i, j\rangle \geq \max\{i, j\} \geq e$ and so R also partially learns f in the sense of Theorem 4. Thus $\mathcal{C}_1 \cup \mathcal{C}_2$ is confidently partially learnable. ∎

Corollary 12. *There is a confidently partially learnable class which is not behaviourally correctly learnble.*

Proof. Blum and Blum's Non-Union Theorem [3] provides classes $\mathcal{C}_1$ and $\mathcal{C}_2$ which are explanatory learnable while their union is not behaviourally correctly learnable. By Theorem 6 the two classes are confidently partially learnable and by Theorem 11 their union $\mathcal{C}_1 \cup \mathcal{C}_2$ is confidently partially learnable as well. ∎

Theorem 3 demonstrates that the class of all total recursive functions is not confidently partially learnable. Nonetheless, there is a less restrictive notion of confident partial learning, somewhat analogous to a blend of behaviourally correct learning and partial learning, that permits the class of all recursive functions to be learnt. This notion of learning is spelt out in the following theorem.

Theorem 13. *There is a recursive learner M such that for all functions f and the sequence of conjectures $e_0, e_1, \ldots$ of M when learning f the following two conditions hold:*

- *There is exactly one partial-recursive function ψ_f for which there are infinitely many n with $\psi_f = \varphi_{e_n}$;*
- *If f is recursive then the ψ_f defined in the first item is equal to f.*

Proof. The learner M works in stages n which are executed in parallel (as some simulations might provide additional indices which have to be taken into account on a stage): M first searches for the first e_n found such that for all $m < n$ it holds that $e_n \geq e_m$ and $\varphi_{e_n}(x) \downarrow = f(x)$ for all $x < n$. From then onwards, the learner searches all $d \leq e_n$ with $\forall x < n\, [\varphi_d(x) \downarrow = f(x)]$; for each such d it outputs the index d itself and a further index $h(d, n, f(n))$ where $\varphi_{h(d,n,f(n))} = \varphi_c$ for the first $c \leq d$ found such that $\forall x \leq n\, [\varphi_c(x) \downarrow = f(x)]$; if such a c does not exist then $\varphi_{h(d,n,f(n))}$ is everywhere undefined.

In the case that e is the least index of f, it follows that M outputs only finitely often an index of the type d or $h(d, n, f(n))$ with $d < e$. M will infnitely often output e. Furthermore, for almost all n, each index of the form $h(d, n, f(n))$ output by M satisfies that $d \geq e$ and that therefore $\varphi_{h(d,n,f(n))} = \varphi_c$ for some φ_c extending $f(0) \circ f(1) \circ \ldots \circ f(n)$. Also, the indices of the form d with $d \neq e$ issued at stage n satisfy that φ_d coincides with f strictly below n. Therefore, the learner issues for each partial function different from f only finitely often an index.

In the case that f is not recursive, then the sequence $e_0, e_1, \ldots$ is increasing and unbounded. For each e_m there is a maximal $n > m$ such that M outputs an index $h(d, n, f(n))$ with $d \leq e_m$. Then $\varphi_{h(d,n,f(n))}$ is the everywhere undefined function, as there is no φ_c with $c \leq n$ such that φ_c extends $f(0) \circ f(1) \circ \ldots \circ f(n)$. Hence M outputs infinitely often an index of the everywhere undefined function. Furthermore, there is no other partial function for which M infinitely often outputs an index: whenever M outputs an index for it at stage n then the corresponding partial function is defined and equal to $f(x)$ at every input $x < n$; as no partial-recursive function coincides with f, M only finitely often outputs an index of that partial function. This completes the proof. ∎

The remainder of the present section is devoted to the study of confident partial learning relative to oracles. As a first step towards characterising the Turing degrees of oracles relative to which all recursive functions can be confidently partially learnt, one may observe that, since the class $\mathcal{C}$ in the proof of Theorem 3 is confidently partially learnable only with respect to high$_2$ oracles, one has the following corollary.

Theorem 14. *There is a behaviourally correctly learnable class $\mathcal{C} \subseteq REC_{0,1}$ such that $\mathcal{C}$ is confidently partially learnable only relative to high$_2$ oracles.*

The next lemma, in whose proof the padding property of the default hypothesis space $\{\varphi_0, \varphi_1, \varphi_2, \ldots\}$ is pivotal, will be applied in the subsequent theorem.

Lemma 15. *For every A''-recursive function $F^{A''}$, there is an A-recursive function f^A such that for all numbers d, if $F^{A''}(d) = e$, then there is a unique number e' for which there are infinitely many t with $f^A(d,t) = e'$ and $\varphi_e = \varphi_{e'}$.*

Proof. Given that $F^{A''} \leq_T A''$, there exists a sequence of A-recursive approximations $\{f_{i,j}\}_{i,j \in \mathbb{N}}$ such that for all numbers e, $\exists i \forall i' \geq i \exists j \forall j' \geq j [f_{i,j}(e) = F^{A''}(e)]$ holds. One may define an A-recursive function G which satisfies $G(e,t) = pad(e', \langle i, s \rangle)$ for some i, s and infinitely many t iff $F^{A''}(e) = e'$.

First, let $a_{e,0}, a_{e,1}, a_{e,2}, \ldots$ be an A-recursive sequence in which $pad(d,i)$ occurs at least n times iff for all $i' \in \{i, i+1, \ldots, i+n\}$, there are n numbers j' such that $f_{i',j'}(e) = d$. This condition ensures that $pad(d,i)$ occurs in $a_{e,0}, a_{e,1}, a_{e,2}, \ldots$ infinitely often for some i iff $d = F^{A''}(e)$; however, the i is not unique and there might be $i' > i$ such that also $pad(d,i')$ also occurs infinitely often in the sequence.

Second, let $a'_{e,0}, a'_{e,1}, a'_{e,2}, \ldots$ be a further A'-recursive sequence in which $pad(d, \langle i, s \rangle)$ occurs n times iff there is a stage $t \geq s$ such that there are n numbers $u \leq s$ with $a_{e,u} = pad(d,i)$ and s is the least number with $pad(d,i') \notin \{a_{e,s}, a_{e,s+1}, \ldots, a_{e,t}\}$ for all $i' < i$.

Subsequently, one may produce the two-valued A-recursive function G by setting $G(e,t) = a'_{e,t}$ for all such sequences $a'_{e,0}, a'_{e,1}, a'_{e,2}, \ldots$ constructed for each e. By the above construction, the A-recursive function G satisfies the condition that for all e, there is exactly one index e' with $G(e,t) = e'$ for infinitely many t and this e' is of the form $pad(F^{A''}(e), \langle i, s \rangle)$ for some i, s. This establishes the lemma. ∎

Having established a necessary condition on the computational power of confident learners that can learn REC, one may hope for an analogous sufficient condition. By means of the above lemma, the theorem below proposes several oracle conditions that, when taken together, enable REC to be confidently partially learnt.

Theorem 16. *If B is low, PA-complete and $A \geq_T B$, $A'' \geq_T \mathbb{K}''$, then there is an A-recursive confident partial learner for REC.*

Proof. First it is shown that the class of all recursive $\{0,1\}$-valued functions, $REC_{0,1}$, is explanatorily learnable by a B-recursive learner which outputs B-recursive indices. For this goal and using that B is PA-complete, one may construct a numbering $\{\varphi^B_{h(0)}, \varphi^B_{h(1)}, \ldots\}$ of total $\{0,1\}$-valued B-recursive functions such that

- $\forall e, x, y, z\, [\varphi^B_{g(e)}(x,y) = 1 \wedge \varphi^B_{g(e)}(x,z) = 1 \Rightarrow y = z]$;
- $\forall e, x, y\, [\varphi_e(x) \downarrow = y \Rightarrow \varphi^B_{g(e)}(x,y) = 1]$.

Due to adequate use of padding in the construction of g, one can assume that $g(e) \geq e$ for all e. There is an B-recursive explanatory learner which conjectures on the input $f(0) \circ f(1) \circ \ldots \circ f(n)$ the index $g(e)$ for the least e with $\varphi^B_{g(e)}(x, f(x)) = 1$ for all $x \leq n$; let M be an equivalent B-recursive confident partial learner and let $g(d_0), g(d_1), g(d_2), \ldots$ be the hypotheses issued by M when it is learning some $f \in REC$.

Second it is shown how to transform this M into the desired A-recursive confident partial learner N. Define the B'''-recursive function $F^{B'''}$ by

$$F^{B'''}(g(d_k)) = \begin{cases} e \text{ if } e \text{ is the minimal index with } graph(\varphi_e) = \varphi^B_{g(d_k)}; \\ 0 \text{ if there is no index } e \text{ with } graph(\varphi_e) = \varphi^B_{g(d_k)}. \end{cases}$$

Furthermore, since $B''' \leq_T A''$ by assumption, it follows that $F^{B'''} = F^{A''}$. One can now define an A-recursive confident partial learner N: by Lemma 15, there is an A-recursive function $f^A(d,t)$ such $f^A(d,t)$ outputs a unique index e' with $\varphi_{e'} = \varphi_{F^{A''}(d)}$ for infinitely many t. When M outputs its k-th hypothesis $g(d_k)$, then N determines $t = |\{k' \leq k : g(d_{k'}) = g(d_k)\}|$ and outputs $pad(f^A(g(d_k), t), g(d_k))$. As M outputs one index d infinitely often, it follows that N outputs exactly one index infinite often and this index is of the form $pad(e', d)$ for some e' satisfying $\varphi^B_{g(d)} = graph(\varphi'_e)$ whenever $\varphi^B_{g(d)}$ is recursive. Furthermore, e' satisfies $\varphi_{pad(e',d)} = f$ whenever f is recursive. ∎

The condition that the double jump of the oracle be Turing above $\mathbb{K}''$ is not, however, sufficient for confidently partially learning REC, as the following theorem demonstrates.

Theorem 17. *There is a set A with $A'' \geq_T \mathbb{K}''$ such that A is 2-generic and $REC_{0,1}$ is not confidently partially learnable relative to A.*

Proof. The proof of this result is based on the existence of a 2-generic set A such that $\mathbb{K}'' \leq_T \mathbb{K}' \oplus A$, so that A is $high_2$, that is, $A'' \geq_T \mathbb{K}''$. It shall be shown that $REC_{0,1}$ is not confidently partially learnable relative to any such set A. Fix such a set A, as well as a $\{0,1\}$-valued total function f which is 2-generic relative to A; one then has that $A \oplus \{\langle x,y \rangle : y = f(x)\}$ is also 2-generic.

Assume towards a contradiction that M^A were a confident partial learner of $REC_{0,1}$. By the confidence of M^A, it must output some index, say e, infinitely often on the sequence for f, where f was chosen as above. Now the following claim is shown.

Claim 18. *There are prefixes α of $A(0) \circ A(1) \circ A(2) \circ \ldots$ and σ of $f(0) \circ f(1) \circ f(2) \ldots$ for which $\forall\beta\forall\tau\exists\gamma\exists\eta[M^{\alpha\circ\beta\circ\gamma}(\sigma \circ \tau \circ \eta) = e]$ holds.*

Now it is shown that this property of M^A follows from the 2-genericity of $A \oplus \{\langle x,y \rangle : y = f(x)\}$. For a proof by contradiction, assume that the prefixes α, σ do not exist and consider the following co-r.e. set W of binary strings:

$$W = \{\beta \oplus \theta : \forall\gamma \in \{0,1\}^*\forall\tau \in \mathbb{N}^*\forall x,y,z[\theta \in \{0,1\}^* \wedge |\theta| = |\beta| \\ \wedge\ (\theta(\langle x,y \rangle) = \theta(\langle x,z \rangle) = 1 \Leftrightarrow y = z)$$

$$\begin{aligned}&\wedge\ ((\max(\{p : \exists q\,[\langle p,q\rangle < |\beta|]\}) < |\tau|\\&\wedge\ (\tau(x) = y \Leftrightarrow \theta(\langle x,y\rangle) = 1))\\&\Rightarrow\ \ (M^{\beta\circ\gamma}(\tau) \neq e))]\},\end{aligned}$$

where the join of two strings $\beta \oplus \theta$ is defined to be the string ξ of length $2\max(|\beta|, |\theta|)$ such that $\xi(2x) = \beta(x)$, $\xi(2x{+}1) = \theta(x)$ whenever $\beta(x), \theta(x)$ are defined; otherwise, $\xi(2x) = \xi(2x{+}1) = 0$. By assumption, for all m, n there exist extensions $A[n] \circ \beta$ and $f[m] \circ \tau$ of $A[n]$ and $f[m]$ respectively such that for any strings $\gamma \in \{0,1\}^*, \eta \in \mathbb{N}^*$, $M^{A[n]\circ\beta\circ\gamma}(f[m] \circ \tau \circ \eta) \neq e$. The constant m and string τ may be chosen so that

$$\max(\{p : \exists q[\langle p,q\rangle < |A[n] \circ \beta|]\}) < |f[m] \circ \tau|,$$

implying that $(A[n] \circ \beta) \oplus \theta \in W$, where θ is a binary string of length $|A[n] \circ \beta|$ with $\theta(\langle x,y\rangle) = 1$ iff $y = (f[m] \circ \tau)(x)$ and $\theta(\langle x,y\rangle) = \theta(\langle x,z\rangle) = 1$ iff $y = z$. Moreover, there cannot exist an n such that, if θ is a binary string of length $n+1$ representing the characteristic function of the set $\{\langle x,y\rangle \leq n : y = f(x)\}$, then $A[n] \oplus \theta \in W$. For, by the hypothesis that M^A outputs e infinitely often on the sequence for f, there must exist $\beta \in \{0,1\}^*$ and $\tau \in \mathbb{N}^*$ satisfying

- $\max(\{p : \exists q[\langle p,q\rangle < |A[n]|]\}) < |\tau|$,
- $\tau(x) = y$ iff $\theta(\langle x,y\rangle) = 1$ and
- $M^{A[n]\circ\beta}(\tau) = e$;

this would thus contradict the condition for $A[n] \oplus \theta$ to be in W. The preceding two conclusions contradict the 2-genericity of $A \oplus \{\langle x,y\rangle : y = f(x)\}$, which means that the prefixes α and σ with the required properties must exist. This completes the proof of the claim; from now on, fix the two prefixes α and σ.

The proof of the theorem proceeds next by constructing two different $\{0,1\}$-valued recursive functions, f_0 and f_1, such that M^A outputs e infinitely often on the sequences for f_0 and f_1. Let f_0 and f_1 be defined as follows.

- At the initial stage, put $f_0(x) = \sigma(x)$ for all $x < |\sigma|$, and $f_0(|\sigma|) = 0$; $f_1(x) = \sigma(x)$ for all $x < |\sigma|$, and $f_1(|\sigma|) = 1$. Let $\sigma_{0,0} = \sigma \circ 0$ and $\sigma_{1,0} = \sigma \circ 1$.
- At stage $s+1$, consider all 2^{s+1} binary strings of length $s+1$; call them $\beta_0, \beta_1, \ldots, \beta_{2^s}$. Search for a sequence of binary strings $\tau_{0,s,0}, \tau_{0,s,1}, \ldots, \tau_{0,s,2^{s+1}}$ with $\tau_{0,s,0} = \sigma_{0,s}$, and for $k = 0,1,\ldots,2^s$, $\tau_{0,s,k+1}$ is a proper extension of $\tau_{0,s,k}$ such that $M^{\alpha\circ\beta_k\circ\gamma_k}(\tau_{0,s,k+1}) \downarrow= e$ for some $\gamma_k \in \{0,1\}^*$. Similarly, find a sequence of binary strings $\tau_{1,s,0}, \tau_{1,s,1}, \ldots, \tau_{1,s,2^{s+1}}$ with $\tau_{1,s,0} = \sigma_{1,s}$, and for $k = 0,1,\ldots,2^s$, there is a $\delta_k \in \{0,1\}^*$ such that $\tau_{1,s,k} \prec \tau_{1,s,k+1}$ and $M^{\alpha\circ\beta_k\circ\delta_k}(\tau_{1,s,k+1}) \downarrow= e$. Let $\sigma_{0,s+1} = \tau_{0,s,2^{s+1}}$ and $\sigma_{1,s+1} = \tau_{1,s,2^{s+1}}$. By the properties of α and σ, the chains of string extensions $\{\tau_{0,s,1}, \tau_{0,s,2}, \ldots, \tau_{0,s,2^{s+1}}\}$, $\{\tau_{1,s,1}, \tau_{1,s,2}, \ldots, \tau_{1,s,2^{s+1}}\}$, as well as the strings γ_k, δ_k must exist, since it may be assumed inductively that σ is a prefix of both $\tau_{0,s,k}$ and $\tau_{1,s,k}$ for $k = 0,1,\ldots,2^s$.
  Set $f_0(x) = \sigma_{0,s+1}(x)$ for all $x \in \mathrm{dom}(\sigma_{0,s+1})$ if $f_0(x)$ is not already defined. Likewise, set $f_1(x) = \sigma_{1,s+1}(x)$ for all $x \in \mathrm{dom}(\sigma_{1,s+1})$ if $f_1(x)$ has not been defined.

It shall be shown that for infinitely many s and binary strings γ_k found in the algorithm at stage $s+1$,

$$(\alpha \circ \beta_k \prec A(0) \circ A(1) \circ A(2) \circ \ldots) \Rightarrow (\alpha \circ \beta_k \circ \gamma_k \prec A(0) \circ A(1) \circ A(2) \circ \ldots).$$

Assume for the sake of a contradiction that there is an s_0 such that for all stages $s+1 > s_0$, whenever $\alpha \circ \beta_k \prec A(0) \circ A(1) \circ A(2) \circ \ldots$, then the string γ_k found with $M^{\alpha\circ\beta_k\circ\gamma_k}(\tau_{0,s,k+1}) \downarrow= e$ fails to satisfy the condition that $A(0) \circ A(1) \circ A(2) \circ \ldots \succ \alpha \circ \beta_k \circ \gamma_k$. Consider the Σ_1^0 set U consisting of all binary strings $\alpha \circ \beta_k \circ \gamma_k$ such that γ_k is the first string found at stage $s+1$ for which $M^{\alpha\circ\beta_k\circ\gamma_k}(\tau_{0,s,k+1}) \downarrow= e$. For all n, there is a stage $s+1 > s_0$ at which $\alpha \circ \beta_k = A(0) \circ A(1) \circ A(2) \circ \ldots \circ A(n)$ for some β_k, and by assumption the string $\alpha \circ \beta_k \circ \gamma_k$ in U is not a prefix of $A(0) \circ A(1) \circ A(2) \circ \ldots$; this contradicts the 2-genericity of A. Hence there are infinitely many stages s at which $M^{A(0)\circ A(1)\circ\ldots\circ A(k)}(\tau_{0,s,n}) = e$ for some numbers k, n, and so M outputs e infinitely often on the sequence for f_0 when it has access to the oracle A. An argument exactly analogous to the preceding one, with δ_k in place of γ_k and $\tau_{1,s,k+1}$ in place of $\tau_{0,s,k+1}$, establishes that M, with access to the oracle A, also outputs e infinitely often on the sequence for f_1. These two conclusions contradict the fact that M must confidently partially learn both the recursive functions f_0 and f_1, since f_0 and f_1 differ on the argument $|\sigma|$, and yet M outputs the same index infinitely often on their respective sequences. In conclusion, $REC_{0,1}$ is not confidently partially learnable relative to A. ∎

5 Essentially class consistent partial learning

The present section considers a weakened form of consistency in partial learning, namely, *essential class consistency*. Under this learning requirement, the learner is permitted to be inconsistent on finitely many segments of the sequence for some recursive function in the class to be learnt. Before developing this notion, we shall first review the more restrictive type of *consistent* learning, and attempt to compare it with confident partial learning.

Definition 19 (Bārzdiņš [2]). A recursive learner M is said to be *consistent* on a total function f iff for all $n \in \mathbb{N}$, $M(f[n]) \downarrow$ and $\varphi_{M(f[n])}(x) \downarrow= f(x)$ whenever $x \leq n$. A learner is said to *class consistently partially* learn $\mathcal{C}$ iff it partially learns $\mathcal{C}$ and is consistent on each f in $\mathcal{C}$.

Whilst class consistency may appear to be a fairly restrictive learning constraint, the following theorem implies that it cannot in general guarantee that a class of recursive functions is confidently partially learnable.

Theorem 20. *There is a class of recursive functions which is class consistently partially learnable but not confidently partially learnable.*

Proof. The following example essentially modifies the construction of the programme $g(d)$ in Theorem 3 so that a subclass of $\mathcal{C}$ may be class consistently partially learnable. For each number d, let $g(d)$ be a programme for a partial-recursive function $\varphi_{g(d)}$ which is defined as follows.

– Set $\varphi_{g(d),s}(0) = d$ for all s.
– Initialize the markers $a_0, a_1, a_2, \ldots$ by setting $a_{i,0} = \langle i, 0\rangle + 1$ for $i \in \mathbb{N}$.
– At stage $s+1$, consider each marker $a_{i,s} = \langle i, r\rangle + 1$ such that $a_{i,s} \le s+1$, and execute the following instructions in succession. Set $\varphi_{g(d),s+1}(x) = 0$ for all $x = \langle i, j\rangle + 1 \le s+1$ such that $j \ne r$ if $\varphi_{g(d),s}$ is not already defined on x. Next, check whether $\varphi_{i,s+1}(a_{i,s}) \downarrow \in \{0, 1\}$ holds; if so, let $\varphi_{g(d),s+1}(a_{i,s}) = 1 - \varphi_{i,s+1}(a_{i,s})$ if $\varphi_{g(d)}$ is not already defined on the input $a_{i,s}$. Now, for each i such that $\langle i, m\rangle + 1 \le s+1$ for some m, let $u = \max(\{m : \langle i, m\rangle + 1 \le s+1\})$. Associate the marker $a_{i,s+1}$ with $\langle i, u+1\rangle + 1$ if at least one of the following two conditions applies; otherwise, let $a_{i,s+1} = a_{i,s}$.
  1. There is a $j < i$ with $\langle j, m\rangle + 1 \le s+1$ for some m such that $a_{j,s+1} \ne a_{j,s}$.
  2. If $a_{i,s} = \langle i, r\rangle + 1$, then the inequality $|\{0, 1, \ldots, r\} - W_{d,s+1}| < i$ holds.

Let $\mathcal{C} = \{f : W_d$ is cofinite $\wedge f$ is a total recursive extension of $\varphi_{g(d)}\}$.

Now it is shown that $\mathcal{C}$ is class consistently partially learnable. First, define a recursive learner N as follows. On input $\sigma = d \circ f(1) \circ \ldots \circ f(n)$, N first identifies the maximum i, if it exists, such that $a_{j,n} = a_{j,n+1}$ for all $j \le i$. If no such i exists, N outputs an index for a partial-recursive function ϕ such that $\phi(x) = f(x)$ for all $x \le n$, and $\phi(x) \uparrow$ for all $x > n$. Otherwise, it conjectures the programme e for which

$$\varphi_e(x) = \begin{cases} f(m) & \text{if } \exists t[m = \langle k, t\rangle + 1 \le n \wedge \varphi_{g(d),n}(m) \uparrow \text{ and } k \le i]; \\ \varphi_{g(d)}(x) & \text{otherwise.} \end{cases}$$

Suppose that N processes a sequence for some recursive function $f \in \mathcal{C}$, so that $W_{f(0)}$ is cofinite. Consider an input sequence $\sigma = d \circ f(1) \circ \ldots \circ f(n)$. If there is a least i such that $a_{i,n} \ne a_{i,n+1}$ and $\langle i, m\rangle + 1 \le n$ for some m, then by condition 1. above, all markers $a_{j,n}$ with $j \ge i$ and $\langle j, l\rangle + 1 \le n$ for some l will be moved to a new position $\langle j, u\rangle + 1$ for which $u = \max\{m : \langle i, m\rangle + 1 \le n+1\}$. Hence $\varphi_{g(d)}$ will be defined on all inputs $\langle j, m\rangle + 1 \le n$ such that $j \ge i$. This in turn implies that N is class consistent.

Next, one shows that N has the following learning characteristic: it outputs incorrect indices only finitely often, and it outputs at least one correct index infinitely often. Let $\sigma = d \circ f(1) \circ \ldots \circ f(n)$ with $i = \max\{j : \forall k \le j[a_{j,n} = a_{j,n+1}]\}$ be a given input sequence. For a case distinction, suppose first that $i > |\overline{W}_d|$. Then, since $W_{g(d)}$ is cofinite and $\varphi_{g(d)}$ is undefined only for values of the form $\langle j, m\rangle + 1$ with $j \le |\overline{W}_d| < i$, there is a sufficiently large stage after which N patches all the undefined places of $\varphi_{g(d)}$ with the correct values of the input function. Secondly, suppose that $i \le |\overline{W}_d|$. As was demonstrated above, each of the markers a_j with $j \le |\overline{W}_d|$ is fixed after a large enough number of computation steps; whence, from this stage onwards, $i \ge |\overline{W}_d|$. Since the marker a_j with $j = |\overline{W}_d| + 1$ moves infinitely often, one concludes that i must be equal to $|\overline{W}_d|$ at infinitely many stages. This establishes the learning property of N claimed at the beginning.

Finally, a class consistent learner M may be built from N as follows: whenever N outputs the sequence of conjectures $e_0, e_1, e_2, \ldots, e_n, \ldots$, M, for each e_n, outputs the index $pad(e_n, k_n)$, where $k_n = |\{m \le n : e_m < e_n\}|$. Then M outputs exactly one correct index for the input function infinitely often, and it is also class consistent. In conclusion, $\mathcal{C}$ is class consistently partially learnable. An argument exactly analogous to that in Theorem 3 shows that this class is not confidently partially learnable. ∎

Essentially class-consistent learners can finitely often be inconsistent with the input sequence; in partial learning, this consistency requirement is still a proper restriction. The following result establishes a connection between the learning success criteria of semantic convergence in the limit and essentially class consistent partial convergence; it suggests that there may be a wealth of examples of essentially class consistently partially learnable classes of recursive functions.

Theorem 21. *Every behaviourally correctly learnable class of recursive functions is essentially class consistently partially learnable.*

Proof. Let $\mathcal{C}$ be a class of recursive functions which is behaviourally correctly learnt by a learner M. Next, define a recursive learner N as follows. On an input f, simulate the learner M and observe the conjectures $e_0, e_1, e_2, \ldots$ output by M. N then outputs a conjecture e_i of M at least s times iff $\forall x \leq s[\varphi_{e_i,s}(x) \downarrow = f(x)]$ holds. If N is presented with the sequence for some $f \in \mathcal{C}$, then M, being a behaviourally correct learner of $\mathcal{C}$, will output only finitely many incorrect indices. Therefore N will output each correct index infinitely often, and every incorrect index finitely often. Now one can build a further learner P: whenever N, on the input sequence, conjectures the sequence $d_0, d_1, d_2, \ldots$, P, for each d_n, outputs $pad(d_n, k_n)$, where $k_n = |\{m \leq n : d_m < d_n\}|$. This learner P is then the required essentially class consistent partial learner of $\mathcal{C}$. ∎

Remark 22. Jain and Stephan [12, Theorem 15] constructed a consistently partially learnable class of recursive functions which is not behaviourally correctly learnable. It follows that the converse of the preceding theorem does not hold in general. Furthermore, they showed [12, Theorem 19] that there are classes which are explanatorily learnable with at most one mind change as well as class consistently explanatorily learnable by a partial-recursive learner, but nonetheless cannot be class consistently partially learnt on canonical texts. Consequently, Theorem 21 is no longer true if one replaces essential class consistency with general class consistency in the conclusion, and so this watered-down variant of consistency is indeed a more general learning notion than ordinary consistency.

To ascertain that essential class consistency constitutes a real learning constraint, one can show that the class of all $\{0, 1\}$-valued recursive functions is not partially learnable under this criterion.

Theorem 23. *The class $REC_{0,1}$ is not essentially class consistently partially learnable.*

One can take the above result one step further and construct an example of a confidently partially learnable class of recursive functions which is not essentially class consistently partially learnable. Note that this class could easily be recoded to be contained in $REC_{0,1}$ in order to prove the above result.

Theorem 24. *There is a class of recursive functions which is confidently partially learnable but not essentially class consistently partially learnable.*

Proof. Let $M_0, M_1, M_2, \ldots$ be a recursive enumeration of all partial-recursive learners.

For each M_e define a function $\varphi_{g(e)}$ by starting with $\sigma_{e,0} = e$ and taking $\sigma_{e,k+1}$ to be the first extension of $\sigma_{e,k}$ found such that $M_e(\sigma_{e,k+1})$ outputs an index d with $\varphi_d(x) \downarrow\neq \sigma_{e,k+1}(x)$ for some $x < |\sigma_{e,k+1}|$. $\varphi_{g(e)}(x)$ takes as value $\sigma_{e,k}(x)$ for the first k found where this is defined.

Furthermore, for each e, k where $\sigma_{e,k}$ is defined, let $\varphi_{h(e,k)}$ be the partial recursive function ψ extending $\sigma_{e,k}$ such that for all $x \geq |\sigma_{e,k}|$, $\psi(x)$ is the least a such that either $M_e(\psi(0)\psi(1)\ldots \psi(x-1)a) > x$ or $M_e(\psi(0)\psi(1)\ldots\psi(x-1)a) = M_e(\psi(0)\psi(1)\ldots\psi(x-1)b)$ for some $b < a$.

Let $\mathcal{C}_1$ contain all those $\varphi_{g(e)}$ which are total and $\mathcal{C}_2$ contain all $\varphi_{h(e,k)}$ where M_e is total and $\varphi_{g(e)} = \sigma_{e,k}$, that is, the construction got stuck at stage k. The class $\mathcal{C}_1$ is obviously explanatorily learnable; for the class $\mathcal{C}_2$, an explanatory learner identifies first the e and then simulates the construction of $\varphi_{g(e)}$ and updates the hypothesis always to $h(e,k)$ for the largest k such that $\sigma_{e,k}$ has already been found. Hence both classes are explanatorily learnable, hence their union $\mathcal{C}$ is confidently partially learnable.

However $\mathcal{C}$ is not essentially class consistently partially learnable, as it is now shown. So consider a total learner M_e. If $\varphi_{g(e)}$ is total then M_e is inconsistent on this function infinitely often and so M_e does not essentially class consistently partially learn $\mathcal{C}$. So consider the k with $\varphi_{g(e)} = \sigma_{e,k}$. Note that the inductive definition of $\varphi_{h(e,k)}$ results in a total function. If M_e outputs on $\varphi_{h(e,k)}$ each index only finitely often, then M_e does not partially learn $\varphi_{h(e,k)}$. If M_e outputs an index d infinitely often, then for all sufficiently long $\tau a \preceq \varphi_{h(e,k)}$ with $M_e(\tau a) = d$ it holds that there is a $b < a$ with $M(\tau b) = d$ as well. By assumption, $\sigma_{e,k+1}$ does not exist and can be neither τa nor τb. Hence τa is not extended by φ_d and so M_e outputs an inconsistent index for almost all times where it conjectures d; again M_e does not essentially class consistently partially learn $\mathcal{C}$. ∎

As a consequence of the proof of the preceding theorem, one has the corollary that essentially class consistent partial learning is not closed under finite unions.

Corollary 25. *Essentially class consistent learning is not closed under finite unions; that is, there are essentially class consistently partially learnable classes $\mathcal{C}_1, \mathcal{C}_2$, such that $\mathcal{C}_1 \cup \mathcal{C}_2$ is not essentially class consistently partially learnable.*

A complete characterisation of the classes of recursive functions which are consistently partially learnable relative to an oracle A, classified according to whether A has hyperimmune or hyperimmune-free Turing degree, was obtained in [12]. The theorem below asserts that a recursive learner with access to a PA-complete oracle may essentially class consistently partially learn REC. Since the class of hyperimmune-free, PA-complete degrees is nonempty, as demonstrated in [13], one may conclude that for partial learning, essential class consistency is indeed a weaker criterion than general consistency, even when learning with oracles. The proof utilises the fact that there is a one-one numbering of all recursive functions plus all functions of finite domain.

Theorem 26. *If A is a PA-complete set, then REC is essentially class consistently partially learnable using A as an oracle.*

Proof. Let $\psi_0, \psi_1, \psi_2, \ldots$ be a one-one numbering of the recursive functions plus the functions with finite domain. For example, Kummer [15] provides such a numbering. Let g be a recursive function such that $\psi_e = \varphi_{g(e)}$ for all e. There is a recursive sequence $(e_0, x_0, y_0), (e_1, x_1, y_1), \ldots$ of pairwise distinct triples such that $\psi_e(x) \downarrow = y$ iff the triple (e, x, y) appears in this sequence.

On input $\sigma = f(0) \circ f(1) \circ \ldots \circ f(n)$, the learner M searches for the first $s \geq n$ such that for all $t \leq s$ either $e_t \neq e_s$ or $x_t > n$ or $y_t = f(x_t)$; that is, s is the first stage where ψ_{e_s} — to the extent it can be judged from the triples enumerated until stage s — is consistent with σ. Then M determines using the PA-complete oracle an $d \leq e_s$ such that either ψ_d extends σ or there is no $c \leq e_s$ such that ψ_c extends σ; note that in that second case the d provided by the oracle does not need to satisfy any condition beyond $d \leq e_s$. The learner conjectures then $g(d)$ for the index d determined this way.

If now e is the unique ψ-index of the function f to be learnt, then for all sufficiently long inputs σ, the above e_s satisfies $e_s \geq e$ as for each $d < e$ either there are only finitely many triples having d in the first component with all of them appearing before n or there is a $t \leq n$ with $e_t = d \wedge x_t \leq n \wedge y_t \neq f(x_t)$. Hence, the s selected satisfies $e_s \geq e$ and therefore the d provided satisfies that ψ_d extends σ. Furthermore, there are infinitely many n with $e_n = e$ and for those the choice is $s = n$ and, if n is sufficiently large, $d = e$. Hence the learner outputs infinitely often a correct index and almost always an index which is consistent with the input seen so far. ∎

6 Essentially globally consistent partial learning

Recall that an essentially globally consistent partial learner outputs on every function, even a non-recursive one, almost always a hypothesis which is consistent with the data seen so far. This has the following consequence: if an essentially globally consistent partial learner outputs a hypothesis e infinitely often then this hypothesis e is correct. The next results establish the basic properties of this learning notion.

Theorem 27. *Every confidently behaviourally correctly learnable class is essentially globally consistently partially learnable.*

Proof. If a class is confidently behaviourally correctly learnable, then there is by Theorem 9 a learner N which outputs on each function f a sequence $d_0, d_1, d_2, \ldots$ of indices such that there is exactly one index e with $d_n = e$ for infinitely many n; furthermore, the sequence satisfies $\varphi_{d_m} = \varphi_e$ for this e and almost all m. In addition to this, whenever f is in the class to be learnt then $\varphi_e = f$ for the index e mentioned above. One uses now this sequence $d_0, d_1, d_2, \ldots$ of hypotheses to construct an essentially globally consistent partial learner M for the given class:

> On input σ, M checks whether it can compute a d_n from σ not yet output such $\varphi_{d_n}(x)\downarrow = \sigma(x)\downarrow$ for all $x < n$. If so, then $M(\sigma) = d_n$ for the least such n else $M(\sigma)$ is a an index greater than $|\sigma|$ for the function $\sigma \circ 0^\infty$.

This learner has now the following properties:

– if the sequence $d_0, d_1, d_2, \ldots$ converges semantically to f then M copies almost every index d_n eventually and therefore outputs the e with $\exists^\infty n\, [d_n = e]$ infinitely often;
– if the sequence $d_0, d_1, d_2, \ldots$ converges semantically to a partial function different from f then M outputs only finitely often an index of the form d_n;
– all other indices output by M are consistent with the data seen so far and do not interfere with convergence in the sense of partial learning as no index gets infinitely output by M that way.

These properties establish that in either case, whether N learns f or not, the learner M only finitely often outputs an index inconsistent with the data seen so far; hence M is essentially globally consistent. Furthermore, M partially learns every function which N partially learns, hence M is an essentially globally consistent partial learner for the given class. ∎

The next statement is quite obvious and shows that essentially consistent partial learning is more general than confident behaviourally correct learning, as for example the class of all almost everywhere constant functions has a consistent explanatory learner but no confident behaviourally correct learner.

Proposition 28. *Every globally consistent explanatory learner is also an essentially globally consistent partial learner for the same class.*

The notion of essentially globally consistent partial learning is closed under union, which stands in contrast to essentially consistent partial learning.

Theorem 29. *If $\mathcal{C}_1$ and $\mathcal{C}_2$ are both essentially globally class consistently learnable then so is $\mathcal{C}_1 \cup \mathcal{C}_2$.*

Proof. Assume that M_1 and M_2 essentially globally class consistently learn the classes $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Now, on input f, let $e_n = \min\{M_1(f[n]), M_2(f[n])\}$ and let d_n be the number of $m < n$ where $e_m < e_n$. The new learner N conjectures on input $f[n]$ the index $pad(e_n, d_n)$.

First, note that for any function f, for almost all n, $M_1(f[n])$ and $M_2(f[n])$ are both consistent with $f[n]$ and therefore so is φ_{e_n} and $\varphi_{pad(e_n,d_n)}$.

Second, there is a least n such that $e_n = e$ for infinitely many n. Furthermore, let d be the number of n with $d_n < e$ which is finite. One can easily see that $N(f[n]) = pad(e, d)$ for infinitely many n. Furthermore, this is the only index which is output infinitely often.

Third, whenever M_1 or M_2 output some index infinitely often then it holds that e_n is infinitely often below this upper bound. Hence there is a least index e which is output infinitely often by either function and N will then on f output $pad(e, d)$ infinitely often for some d. As N is essentially globally consistent, N then learns f. ∎

The class of Theorem 24 is not essentially class consistently partially learnable but the union of two explanatorily learnable classes. As essentially globally consistent partial learning is more restrictive than essentially class consistent partial learning and furthermore closed under union, one of these two explanatorily learnable classes cannot be essentially globally consistently partially learnable.

Corollary 30. *There is an explanatorily learnable class which is not essentially globally consistently partially learnable.*

7 Conclusion

In conclusion, confident partial learning appears to be a fairly robust learning notion that is neither too restrictive nor too powerful. Essentially class consistent partial learning may be a more balanced criterion compared to global consistency, for there is quite a rich collection of essentially class consistently partially learnable classes of recursive functions, which includes all classes that are behaviourally correctly learnable. Though the results on these two notions are quite complete, there is still potential for further work on characterising the omniscient degrees of inference for confident partial learning and essentially class consistent partial learning.

References

1. Janis Bārzdiņš. Two theorems on the limiting synthesis of functions. In *Theory of Algorithms and Programs, vol. 1*, pages 82–88. Latvian State University, 1974. In Russian.
2. Janis Bārzdiņš. Inductive inference of automata, functions and programs. In *Proceedings of the 20th International Congress of Mathematicians, Vancouver*, pages 455-560, 1974. In Russian. English translation in American Mathematical Society Translations: Series 2, 109: 107–112, 1977.
3. Lenore Blum and Manuel Blum. Towards a mathematical theory of inductive inference. *Information and Control* 28 (1975): 125–155.
4. John Case and Carl Smith. Comparison of identification criteria for machine inductive inference. *Theoretical Computer Science*, 25 (1983): 193–220.
5. Jerome Feldman. Some decidability results on grammatical inference and complexity. *Information and Control* 20 (1972): 244–262.
6. Rusins Freivalds, Efim Kinber and Rolf Wiehagen. Inductive inference and computable one-one numberings. *Zeitschrift fuer mathematische Logik und Grundlagen der Mathematik* 28 (1982): 463–479.
7. Ziyuan Gao, Frank Stephan, Guohua Wu and Akihiro Yamamoto. Learning families of closed sets in matroids. *Computation, Physics and Beyond; International Workshop on Theoretical Computer Science*, WTCS 2012, Springer LNCS 7160 (2012): 120–139.
8. E. Mark Gold. Language identification in the limit. *Information and Control* 10 (1967): 447–474.
9. Gunter Grieser. Reflective inductive inference of recursive functions. *Theoretical Computer Science A* 397(1–3) (2008): 57–69.
10. William Hanf. The Boolean algebra of logic. *Bulletin of the American Mathematical Society* 81 (1975): 587–589.
11. Sanjay Jain, Daniel Osherson, James S. Royer and Arun Sharma. 1999. *Systems that learn: an introduction to learning theory.* Cambridge, Massachusetts.: MIT Press.

12. Sanjay Jain and Frank Stephan. Consistent partial identification. COLT 2009: 135–145.
13. Carl G. Jockusch, Jr and Robert I. Soare. Π^0_1 classes and degrees of theories. *Transactions of the American Mathematical Society* 173 (1972): 33–56.
14. Imre Lakatos. The role of crucial experiments in science. *Studies in History and Philosophy of Science* 4(4) (1974): 319–320.
15. Martin Kummer. Numberings of $\mathcal{R}_1 \cup F$. *Computer Science Logic 1988, Springer Lecture Notes in Computer Science* 385 (1989): 166–186.
16. Daniel N. Osherson, Michael Stob and Scott Weinstein. 1986. *Systems that learn: an introduction to learning theory for cognitive and computer scientists*. Cambridge, Massachusetts.: MIT Press.
17. Karl R. Popper. 1959. *The logic of scientific discovery*. London: Hutchinson.
18. Hartley Rogers, Jr. 1987. *Theory of recursive functions and effective computability*. Cambridge, Massachusetts: MIT Press.
19. Rolf Wiehagen and Thomas Zeugmann. Learning and consistency. *Algorithmic Learning for Knowledge-Based Systems*, GOSLER Final Report, Springer LNAI 961 (1995): 1–24.
20. Thomas Zeugmann and Sandra Zilles. Learning recursive functions: a survey. *Theoretical Computer Science* 397(1-3) (2008): 4–56.